# Quantitative Stability of Barycenters in the Wasserstein Space


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January 2023

# Introduction

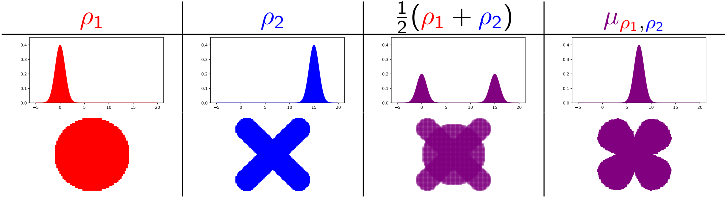
## Wassertein Barycenters

**Definition:** Let $\Omega \subset \mathbb{R}^d$ compact. *Wasserstein barycenter* of $\rho_1, \dots, \rho_N \in \mathcal{P}(\Omega)$:

$$\boxed{\mu_{\rho_1,\dots,\rho_N} \in \arg\min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2N} \sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu),}$$

where $\forall \alpha, \beta \in \mathcal{P}(\Omega), \mathrm{W}_2^2(\alpha, \beta) = \min_{\gamma \in \Gamma(\alpha,\beta)} \int_{\Omega\times\Omega} \|x-y\|^2 \, \mathrm{d}\gamma(x,y)$.

- Geometrically faithful "average" of probability measures:

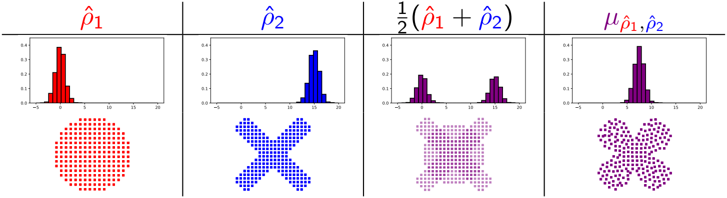


# Introduction

## Wassertein Barycenters

- Many applications, e.g. in
  1. Texture synthesis (Rabin et al., 2011).
  2. Geometry processing (Solomon et al., 2015).
  3. Language processing (Colombo et al., 2021).

- $\rho_1$, $\rho_2$ often **not directly accessible**, but $\hat{\rho}_1$, $\hat{\rho}_2$ instead:




**Can we bound $\mathrm{W}_2(\mu_{\hat{\rho}_1,\hat{\rho}_2}, \mu_{\rho_1,\rho_2})$ in terms of $\mathrm{W}_2(\hat{\rho}_1, \rho_1)$ and $\mathrm{W}_2(\hat{\rho}_2, \rho_2)$?**

# Introduction

## Stability of Wassertein Barycenters - Positive results

- **Consistency:**

**Theorem** (Le Gouic, Loubes, 2017):
If $\forall i, \mathrm{W}_2(\rho_i^n, \rho_i) \xrightarrow[n\to\infty]{} 0$, then $(\mu_{\rho_1^n,\dots,\rho_N^n})_n$ is precompact and any limit is a barycenter of $\rho_1, \dots, \rho_N$.

**Quantitative version?**

# Introduction

## Stability of Wassertein Barycenters - Positive results

- **Quantitative stability in dimension $d = 1$:**

**Proposition**:
In dimension $d = 1$, $\mathrm{W}_2$ is Hilbertian:

$$\mathrm{W}_2(\alpha, \beta) = \left\| F_\alpha^{-1} - F_\beta^{-1} \right\|_{\mathrm{L}^2([0,1])}.$$

As a consequence:

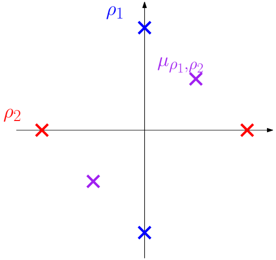
$$\boxed{\mathrm{W}_2(\mu_{\rho_1,\dots,\rho_N}, \mu_{\tilde{\rho}_1,\dots,\tilde{\rho}_N}) \leq \frac{1}{N} \sum_{i=1}^{N} \mathrm{W}_2(\rho_i, \tilde{\rho}_i).}$$

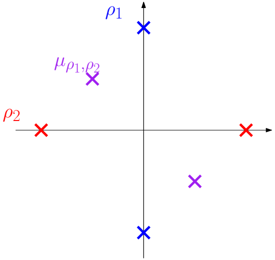
**Quantitative stability result in dimension $d \geq 2$?**

# Introduction

## Stability of Wassertein Barycenters - Negative results

- **When $d > 1$, barycenter may not be unique:**

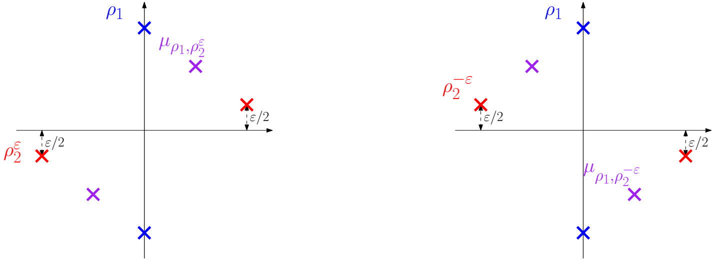


# Introduction

## Stability of Wassertein Barycenters - Negative results

- **No quantitative stability is possible:**



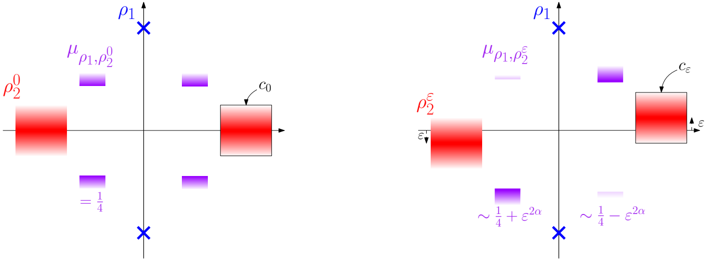

$$W_2(\rho_2^\varepsilon, \rho_2^{-\varepsilon}) = \varepsilon \text{ while } W_2(\mu_{\rho_1,\rho_2^\varepsilon}, \mu_{\rho_1,\rho_2^{-\varepsilon}}) = 1.$$

# Introduction

## Stability of Wassertein Barycenters - Negative results

**Proposition** (Agueh, Carlier, 2011):
If one of the $\rho_i$'s is absolutely continuous, the barycenter is unique.

- Even with an a.c. marginal, $\alpha$-**Hölder behaviour for any** $\alpha \in (0,1)$ **is possible**:




$$\mathrm{W}_2(\rho_2^0, \rho_2^\varepsilon) = \varepsilon \text{ while } \mathrm{W}_2(\mu_{\rho_1,\rho_2^0}, \mu_{\rho_1,\rho_2^\varepsilon}) \sim \varepsilon^\alpha.$$

# Outline

Part I

Main result.

Consequence: plug-in estimation of Wasserstein barycenters.

Part II

Sketch of proof.

Main tool: strong-convexity of the variance functional.

Part III

General result.

Consequence: Statistics in the Wasserstein space.

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# Stability of Wasserstein Barycenters

## Main result

**Theorem** (Carlier, D., Mérigot, 2022):

- Let $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$ and $\tilde{\rho}_1, \ldots, \tilde{\rho}_N \in \mathcal{P}(\Omega)$ such that:
  1. $\rho_1$ **admits a bounded density and satisfies a Poincaré-Wirtinger inequality**: $\exists C_{PW} > 0$ s.t. $\forall f \in \mathcal{C}^1(\Omega)$,
  $$\|f - \langle f|\rho_1\rangle\|_{\mathrm{L}^1(\rho_1)} \leq C_{PW} \|\nabla f\|_{\mathrm{L}^1(\rho_1)} .$$
  2. $\mathrm{spt}(\rho_1)$ is a **connected union of** $K$ **convex sets**.
- Then:

$$\mathrm{W}_2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\tilde{\rho}_1,\ldots,\tilde{\rho}_N}) \leq C_{d,R,\rho_1} N^{1/6} \left( \frac{1}{N} \sum_{i=1}^{N} \mathrm{W}_2(\rho_i, \tilde{\rho}_i) \right)^{1/6} ,$$

where $C_{d,R,\rho_1} = C_d R^{d+1} \mathrm{per}(\mathrm{spt}(\rho_1))^{1/3} \frac{\|\rho_1\|_\infty}{\|1/\rho_1\|_\infty} \frac{K^2}{\varepsilon} C_{PW}$.

- Remarks:
  1. Optimal assumptions?
  2. Optimal exponent?

# Stability of Wasserstein Barycenters

## Statistical consequence

**Theorem** (Fournier, Guillin, 2015):

- Let $\rho \in \mathcal{P}(\Omega)$ and $\hat\rho^n = \frac{1}{n}\sum_{j=1}^n \delta_{x_j}$ where $(x_j)_{1\le j\le n} \sim \rho^{\otimes n}$. Then:

$$\mathbb{E}\mathrm{W}_2^2(\hat\rho^n, \rho) \le C_d R^2 \begin{cases} n^{-1/2} & \text{if } d < 4, \\ n^{-1/2}\log(n) & \text{if } d = 4, \\ n^{-2/d} & \text{else.} \end{cases}$$

**Corollary** (Carlier, D., Mérigot, 2022):

- Under the assumptions of the theorem, if $\forall i$, $\hat\rho_i = \frac{1}{n}\sum_{j=1}^n \delta_{x_{i,j}}$ where $(x_{i,j})_{1\le j\le n} \sim \rho_i^{\otimes n}$, then

$$\boxed{\mathbb{E}\mathrm{W}_2^2(\mu_{\rho_1,\dots,\rho_N}, \mu_{\hat\rho_1^n,\dots,\hat\rho_N^n}) \lesssim N^{1/3} \begin{cases} n^{-1/12} & \text{if } d < 4, \\ n^{-1/12}\log(n)^{1/6} & \text{if } d = 4, \\ n^{-1/(3d)} & \text{else.} \end{cases}}$$

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Sketch of proof.

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General result.

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# Sketch of proof

## Strong-convexity of the variance functional?

**Definition:** *Variance functional* associated to $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$:

$$F_{\rho_1,\ldots,\rho_N} : \mu \mapsto \frac{1}{2N} \sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu).$$

- $F_{\rho_1,\ldots,\rho_N}$ is convex.

Stability estimate for $\mu_{\rho_1,\ldots,\rho_N} \in \arg\min_{\mu\in\mathcal{P}(\Omega)} F_{\rho_1,\ldots,\rho_N}(\mu)$.

$\Longleftrightarrow$

*Strong-convexity* estimate for $F_{\rho_1,\ldots,\rho_N}$.

- $F_{\rho_1,\ldots,\rho_N} = \frac{1}{N}\sum_{i=1}^N f_{\rho_i}$, where $\forall \rho \in \mathcal{P}(\Omega)$, $f_\rho : \mu \mapsto \frac{1}{2}\mathrm{W}_2^2(\rho, \mu).$

**When is $f_\rho$ *strongly-convex*?**

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# Sketch of proof

**Strong-convexity of $f_\rho = \frac{1}{2}\mathrm{W}_2^2(\rho, \cdot)$?**

When is $f_\rho : \mu \mapsto \frac{1}{2}\mathrm{W}_2^2(\rho, \mu)$ *strongly-convex*?

- Kantorovich duality:

$$\frac{1}{2}\mathrm{W}_2^2(\rho, \mu) = \langle \frac{\|\cdot\|^2}{2} | \rho \rangle + \langle \frac{\|\cdot\|^2}{2} | \mu \rangle - \Big( \min_{\psi : \mathbb{R}^d \to \mathbb{R}} \langle \psi^* | \rho \rangle + \langle \psi | \mu \rangle \Big),$$

where $\psi^*(\cdot) = \sup_{y \in \mathbb{R}^d} \langle \cdot | y \rangle - \psi(y)$ is the Legendre transform of $\psi$.

- *Subdifferential* of $f_\rho$:

$$\partial f_\rho(\mu) = \left\{ \frac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} \mid \psi_{\rho\to\mu} \in \arg\min_{\psi} \langle \psi^* | \rho \rangle + \langle \psi | \mu \rangle \right\}.$$

$$\forall \mu, \nu \in \mathcal{P}(\Omega), \quad \langle \frac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu \rangle \leq f_\rho(\nu) - f_\rho(\mu).$$

$\to$ **Gap in this inequality?**

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$$\forall \mu, \nu \in \mathcal{P}(\Omega), \qquad f_\rho(\mu) + \langle \frac{\|\cdot\|^2}{2} - \psi_{\rho \to \mu} | \nu - \mu \rangle \leq f_\rho(\nu).$$

$\rightarrow$ **Gap in this inequality?**

# Sketch of proof

## Strong-convexity of the Kantorovich functional?

Gap in $f_\rho(\mu) + \langle \frac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu \rangle \leq f_\rho(\nu)$?

**Definition:** *Kantorovich functional* associated to $\rho \in \mathcal{P}(\Omega)$ :

$$\mathcal{K}_\rho : \psi \mapsto \langle \psi^* | \rho \rangle.$$

- From Kantorovich duality,

$$f_\rho(\nu) - f_\rho(\mu) - \langle \frac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu \rangle$$

$$=$$

$$\mathcal{K}_\rho(\psi_{\rho\to\mu}) - \mathcal{K}_\rho(\psi_{\rho\to\nu}) - \langle -\nu | \psi_{\rho\to\mu} - \psi_{\rho\to\nu} \rangle.$$

Note that $-\nu \in \partial \mathcal{K}_\rho(\psi_{\rho\to\nu})$

(since $0 \in \partial \mathcal{K}_\rho(\psi_{\rho\to\nu}) + \nu$, since $\psi_{\rho\to\nu} \in \arg\min_\psi \mathcal{K}_\rho(\psi) + \langle \psi | \nu \rangle$)

$\rightarrow$ **When is $\mathcal{K}_\rho$ *strongly-convex*?**

# Sketch of proof

## Strong-convexity of the Kantorovich functional?

When is $\mathcal{K}_\rho : \psi \mapsto \langle \psi^* | \rho \rangle$ *strongly-convex*?

1. *Strong-convexity should work "up to additive constants":*

$$\forall c \in \mathbb{R}, \quad \mathcal{K}_\rho(\psi + c) = \mathcal{K}_\rho(\psi) - c.$$

2. *Support of $\rho$ should be connected:*

**Theorem** (Brenier, 1987): If $\rho$ is **absolutely continuous**, then the optimal transport solution between $\rho$ and any $\mu \in \mathcal{P}(\Omega)$ is **unique** and it is induced by any convex function $\phi$ satisfying $(\nabla\phi)_\# \rho = \mu$.




$$\nabla\psi^*_{\mu\#}\rho = \nabla\tilde{\psi}^*_{\mu\#}\rho = \mu.$$

$$\implies \psi_\mu, \tilde{\psi}_\mu \in \arg\min_\psi \mathcal{K}_\rho(\psi) + \langle \psi | \mu \rangle.$$

$$\implies \forall t \in [0,1],$$

$$\mathcal{K}_\rho((1-t)\psi_\mu + t\tilde{\psi}_\mu) = (1-t)\mathcal{K}_\rho(\psi_\mu) + t\mathcal{K}_\rho(\tilde{\psi}_\mu).$$

**Assumption:** Source $\rho$ is **absolutely continuous** and satisfies a **Poincaré-Wirtinger inequality**: $\exists p \geq 1, C_{PW} \in (0, +\infty)$ s.t.

$$\forall f \in \mathcal{C}^1(\mathbb{R}^d), \quad \|f - \mathbb{E}_\rho f\|_{\mathrm{L}^p(\rho)} \leq C_{PW} \|\nabla f\|_{\mathrm{L}^p(\rho,\mathbb{R}^d)}.$$

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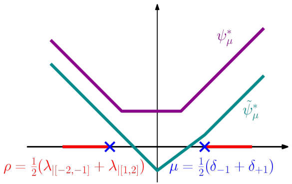


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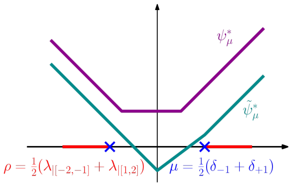


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# Sketch of proof

## Strong-convexity of the Kantorovich functional?

- A known result:

**Theorem** (Ambrosio, Gigli, 2011): Assume $\psi_{\rho\to\mu}$ is $\alpha$-**strongly convex for some** $\alpha > 0$. Then:

$$\frac{\alpha}{2C_{PW}}\mathbb{V}\mathrm{ar}_\rho(\psi^*_{\rho\to\mu} - \psi^*_{\rho\to\nu}) \leq \mathcal{K}_\rho(\psi_{\rho\to\mu}) - \mathcal{K}_\rho(\psi_{\rho\to\nu}) - \langle -\nu | \psi_{\rho\to\mu} - \psi_{\rho\to\nu}\rangle.$$

**Strong assumption:**

$\psi_{\rho\to\mu}$ is $\alpha$-strongly convex $\iff$ $\nabla\psi^*_{\rho\to\mu}$ is $\frac{1}{\alpha}$-Lipschitz continuous!

$\to$ Not satisfied in general.
$\to$ Implies that $(\nabla\psi^*_{\rho\to\mu})_{\#}\rho = \mu$ has a connected support.
$\to$ In our context, $\mu$ is a (candidate) barycenter: no regularity theory.

# Sketch of proof

## Strong-convexity of the Kantorovich functional

**Theorem** (D., Mérigot, 2021):

- Let $\rho \in \mathcal{P}_{a.c.}(\Omega)$ with with bounded density on $\mathrm{spt}(\rho)$ that is assumed to be **convex**.

Then for all $\mu, \nu \in \mathcal{P}(\Omega)$,

$$C_{d,R,\rho} \mathbb{V}\mathrm{ar}_\rho(\psi^*_{\rho\to\mu} - \psi^*_{\rho\to\nu}) \leq \mathcal{K}_\rho(\psi_{\rho\to\mu}) - \mathcal{K}_\rho(\psi_{\rho\to\nu}) - \langle -\nu | \psi_{\rho\to\mu} - \psi_{\rho\to\nu} \rangle,$$

where $C_{d,R,\rho} = \left( e(d+1)2^{d-1} R^2 \frac{\|\rho\|_\infty^2}{\|1/\rho\|_\infty^2} \right)^{-1}$.

- Remarks:
  1. **Proof idea:** lower-bound on $\frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{K}_\rho((1-t)\psi_{\rho\to\mu} + t\psi_{\rho\to\nu})$ from the **Brascamp-Lieb** inequality.
  2. Similar result with **non-compact targets** (moment assumptions).
  3. Optimal exponents.
  4. $\mathrm{spt}(\rho)$ convex? Can be relaxed.

**Corollary** (Carlier, D., Mérigot, 2022): If $\mathrm{spt}(\rho)$ is a **connected finite union of convex sets** s.t. $\rho$ satisfies a $\mathrm{L}^1$ **Poincaré-Wirtinger inequality**, a similar estimate holds.

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Then for all $\mu, \nu \in \mathcal{P}(\Omega)$,

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where $C_{d,R,\rho} = \left( e(d+1)2^{d-1}R^2 \frac{\|\rho\|^2_\infty}{\|1/\rho\|^2_\infty} \right)^{-1}$.

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# Sketch of proof

**Strong-convexity of $f_\rho = \frac{1}{2}\mathrm{W}_2^2(\rho,\cdot)$**

- Recall that

$$\mathcal{K}_\rho(\psi_{\rho\to\mu}) - \mathcal{K}_\rho(\psi_{\rho\to\nu}) - \langle -\nu | \psi_{\rho\to\mu} - \psi_{\rho\to\nu}\rangle$$
$$=$$
$$\tfrac{1}{2}\mathrm{W}_2^2(\rho,\nu) - \tfrac{1}{2}\mathrm{W}_2^2(\rho,\mu) - \langle \tfrac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu\rangle.$$

**Corollary** (D., Mérigot, 2021):

- Let $\rho \in \mathcal{P}_{a.c.}(\Omega)$ with bounded density satisfying a $\mathrm{L}^1$ Poincaré-Wirtinger inequality. Assume that $\mathrm{spt}(\rho)$ is a connected union of $K$ convex sets.

Then for all $\mu, \nu \in \mathcal{P}(\Omega)$,

$$C_{d,R,\rho}\mathrm{W}_2^6(\mu,\nu) \le \frac{1}{2}\mathrm{W}_2^2(\rho,\nu) - \frac{1}{2}\mathrm{W}_2^2(\rho,\mu) - \langle \frac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu\rangle,$$

where $C_{d,R,\rho} = \left(C_d R^{3d+2}\mathrm{per}(\mathrm{spt}(\rho))^2 \frac{\|\rho\|_\infty^5}{\|1/\rho\|_\infty^5} \frac{K^7}{\varepsilon^6} C_{PW}^6\right)^{-1}$.

- $\mathrm{W}_2^6(\mu,\nu) \lesssim \mathbb{V}\mathrm{ar}_\rho(\psi^*_{\rho\to\mu} - \psi^*_{\rho\to\nu})$ obtained from new Galgliardo-Nirenberg type inequality:

**Proposition** (D., Mérigot, 2021): For $K \subset \mathbb{R}^d$ compact and $u, v : K \to \mathbb{R}$ Lipschitz convex,

$$\|\nabla u - \nabla v\|^6_{\mathrm{L}^2(K,\mathbb{R}^d)} \le C_d \mathcal{H}^{d-1}(\partial K)^2 (\mathrm{Lip}(u) + \mathrm{Lip}(v))^4 \|u - v\|^2_{\mathrm{L}^2(K)}.$$

# Sketch of proof

**Strong-convexity of $f_\rho = \frac{1}{2}\mathrm{W}_2^2(\rho,\cdot)$**

- Recall that

$$\mathcal{K}_\rho(\psi_{\rho\to\mu}) - \mathcal{K}_\rho(\psi_{\rho\to\nu}) - \langle -\nu | \psi_{\rho\to\mu} - \psi_{\rho\to\nu}\rangle$$
$$=$$
$$\tfrac{1}{2}\mathrm{W}_2^2(\rho,\nu) - \tfrac{1}{2}\mathrm{W}_2^2(\rho,\mu) - \langle \tfrac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu\rangle.$$

**Corollary** (D., Mérigot, 2021):

- Let $\rho \in \mathcal{P}_{a.c.}(\Omega)$ with bounded density satisfying a $\mathrm{L}^1$ Poincaré-Wirtinger inequality. Assume that $\mathrm{spt}(\rho)$ is a connected union of $K$ convex sets.

Then for all $\mu,\nu \in \mathcal{P}(\Omega)$,

$$\boxed{C_{d,R,\rho}\mathrm{W}_2^6(\mu,\nu) \le \frac{1}{2}\mathrm{W}_2^2(\rho,\nu) - \frac{1}{2}\mathrm{W}_2^2(\rho,\mu) - \langle \frac{\|\cdot\|^2}{2} - \psi_{\rho\to\mu} | \nu - \mu\rangle,}$$

where $C_{d,R,\rho} = \left( C_d R^{3d+2} \mathrm{per}(\mathrm{spt}(\rho))^2 \frac{\|\rho\|_\infty^5}{\|1/\rho\|_\infty^5} \frac{K^7}{\varepsilon^6} C_{PW}^6 \right)^{-1}$.

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# Sketch of proof

## Strong-convexity of the variance functional

- Recall that $F_{\rho_1,\dots,\rho_N}(\mu) = \frac{1}{2N}\sum_{i=1}^N \mathrm{W}_2^2(\rho_i,\mu)$.

**Corollary** (Carlier, D., Mérigot, 2022):
Under the assumptions of the main theorem, for all $\mu,\nu \in \mathcal{P}(\Omega)$,

$$\boxed{\frac{1}{N}\mathrm{W}_2^6(\mu,\nu) \lesssim F_{\rho_1,\dots,\rho_N}(\nu) - F_{\rho_1,\dots,\rho_N}(\mu) - \langle \frac{\|\cdot\|^2}{2} - \frac{1}{N}\sum_i \psi_{\rho_i\to\mu} | \nu - \mu\rangle.}$$

- Remarks:
  1. The stability result follows immediately together with $|F_{\rho_1,\dots,\rho_N}(\cdot) - F_{\tilde\rho_1,\dots,\tilde\rho_N}(\cdot)| \lesssim \frac{1}{N}\sum_i \mathrm{W}_2(\rho_i,\tilde\rho_i)$.
  2. May be used to get stability of *regularized* Wasserstein barycenters.

# Outline

## Part I
## Main result.

Consequence: plug-in estimation of Wasserstein barycenters.

## Part II
## Sketch of proof.

Main tool: strong-convexity of the variance functional.

## Part III
## General result.

Consequence: Statistics in the Wasserstein space.

# Stability - General result

## Wassertein Barycenter - Extension of definition

- Wasserstein barycenter of $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$:

$$\mu_{\rho_1,\ldots,\rho_N} \in \arg \min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2N} \sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu).$$

- Extension 1: weight $\rho_i$ with $\alpha_i > 0$:

$$\boxed{\mu_{\alpha_1\rho_1,\ldots,\alpha_N\rho_N} \in \arg \min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2\sum_i \alpha_i} \sum_{i=1}^{N} \alpha_i \mathrm{W}_2^2(\rho_i, \mu).}$$

- Extension 2: allow $N \to \infty$. Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$:

$$\boxed{\mu_{\mathbb{P}} \in \arg \min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2} \int_{\mathcal{P}(\Omega)} \mathrm{W}_2^2(\rho, \mu) \mathrm{d}\mathbb{P}(\rho).}$$

# Stability - General result

**Theorem** (Carlier, D., Mérigot, 2022):

- Let $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$. Assume that there exists $S_{\mathbb{P}} \subset \mathcal{P}_{a.c.}(\Omega)$ such that $\boxed{\mathbb{P}(S_{\mathbb{P}}) = \alpha_{\mathbb{P}} > 0}$ and $\forall \rho \in S_{\mathbb{P}}$:
  1. $\rho$ **admits a bounded density and satisfies a Poincaré-Wirtinger inequality**.
  2. $\mathrm{spt}(\rho)$ is a finite connected union of convex sets.
- Then:

$$\boxed{\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/6}} \mathcal{W}_1(\mathbb{P}, \mathbb{Q})^{1/6},}$$

and

$$\boxed{\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/5}} \|\mathbb{P} - \mathbb{Q}\|_{\mathrm{TV}}^{1/5}.}$$

$$\forall \mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega)), \quad \mathcal{W}_1(\mathbb{P}, \mathbb{Q}) := \min_{\gamma \in \Gamma(\mathbb{P}, \mathbb{Q})} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathrm{W}_2(\rho, \tilde{\rho}) \mathrm{d}\gamma(\rho, \tilde{\rho}).$$

# Stability - General result

## Statistics in the Wasserstein Space

- Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ and $\mathbb{P}_m = \frac{1}{m}\sum_{i=1}^m \delta_{\rho_i}$ with $(\rho_i)_{1\leq i\leq m} \sim \mathbb{P}^{\otimes m}$.

  $\rightarrow$ How fast does $\mu_{\mathbb{P}_m}$ approximate $\mu_{\mathbb{P}}$ in terms of $m$?

- A known result:

**Theorem** (Le Gouic et al., 2022):
Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ and a barycenter $\mu_{\mathbb{P}}$ s.t. $\forall \rho \in \mathrm{spt}(\mathbb{P}), \rho = (\nabla\psi_{\mu_{\mathbb{P}}\to\rho})_{\#}\mu_{\mathbb{P}}$ with $\boxed{\alpha Id \preceq \mathrm{D}^2\psi_{\mu_{\mathbb{P}}\to\rho} \preceq \beta Id}$. Then if $\beta - \alpha < 1$, $\mu_{\mathbb{P}}$ is unique and

$$\boxed{\mathbb{E}\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m}) \leq \frac{4R}{\sqrt{1-\beta+\alpha}} m^{-1/2}.}$$

- Remarks:
  1. **Parametric rate.**
  2. Very strong assumption: no such regularity theory yet for Wasserstein barycenters.

# Stability - General result

## Statistics in the Wasserstein Space

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# Stability - General result

## Statistics in the Wasserstein Space

- Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ and $\mathbb{P}_m = \frac{1}{m}\sum_{i=1}^m \delta_{\rho_i}$ with $(\rho_i)_{1\leq i\leq m} \sim \mathbb{P}^{\otimes m}$.
- If $\mathbb{P}$ gives mass $\alpha_{\mathbb{P}}$ to a set of "nice" measures, then:

$$\boxed{\mathbb{E}\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/6}}\mathbb{E}\mathcal{W}_1(\mathbb{P}, \mathbb{P}_m)^{1/6}.}$$

- If **upper Wasserstein dimension of** $\mathbb{P} < s$ (Definition 4 of (Weed and Bach (2019))) then,

$$\mathbb{E}\mathcal{W}_1(\mathbb{P}, \mathbb{P}_m) \lesssim m^{-1/s}.$$

- If $\alpha_{\mathbb{P}} = 1$:

**Corollary** (Carlier, D., Mérigot, 2022):

- Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ and $\mathbb{P}_m = \frac{1}{m}\sum_{i=1}^m \delta_{\rho_i}$ with $(\rho_i)_{1\leq i\leq m} \sim \mathbb{P}^{\otimes m}$. Assume that **for all** $\rho \in \mathrm{spt}(\mathbb{P})$,
  1. $\rho$ **admits a bounded density and satisfies a Poincaré-Wirtinger inequality** and $\mathrm{spt}(\rho)$ is a finite connected union of convex sets.
- Then:

$$\boxed{\mathbb{E}\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m}) \lesssim m^{-1/30}.}$$

# Stability - General result

## Statistics in the Wasserstein Space

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Thank you for your attention!

# Appendix

## Local strong convexity of the Kantorovich functional $\mathcal{K}_\rho$

**For $\rho$ with convex support.** Let $\mu^0, \mu^1 \in \mathcal{P}(\Omega)$ and for $k \in \{0,1\}$,

$$\psi^k \in \arg \min_{\psi \in \mathcal{C}(\Omega), \langle \psi | \mu^k \rangle = 0} \mathcal{K}_\rho(\psi).$$

For $t \in [0,1]$ denote $\psi^t = (1-t)\psi^0 + t\psi^1 = \psi^0 + tv$, and notice that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_\rho(\psi^t) = -\mathbb{E}_\rho v(\nabla \psi^{t*}),$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{K}_\rho(\psi^t) = \mathbb{E}_\rho \langle \nabla v(\nabla \psi^{t*}) | \left(\mathrm{D}^2 \psi^t\right)^{-1} \nabla v(\nabla \psi^{t*}) \rangle.$$

Brascamp-Lieb inequality:

$$\boxed{\mathbb{V}\mathrm{ar}_\rho(v(\nabla \psi^{t*})) \lesssim \mathbb{E}_\rho \langle \nabla v(\nabla \psi^{t*}) | \left(\mathrm{D}^2 \psi^t\right)^{-1} \nabla v(\nabla \psi^{t*}) \rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{K}_\rho(\psi^t).}$$

# Appendix

**Local strong convexity of the Kantorovich functional $\mathcal{K}_\rho$**

Brascamp-Lieb inequality:

$$\boxed{\mathbb{V}\mathrm{ar}_\rho(v(\nabla\psi^{t*})) \lesssim \mathbb{E}_\rho\langle \nabla v(\nabla\psi^{t*})| \left(\mathrm{D}^2\psi^t\right)^{-1} \nabla v(\nabla\psi^{t*})\rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{K}_\rho(\psi^t).}$$

$\int_0^1 \dots \mathrm{d}t+$ concavity of $A \mapsto \det(A)^{1/d}$:

$$\mathbb{V}\mathrm{ar}_{\frac{\mu^0+\mu^1}{2}}(\psi^1 - \psi^0) \lesssim \mathcal{K}_\rho(\psi^1) - \mathcal{K}_\rho(\psi^0) - \langle \nabla\mathcal{K}_\rho(\psi^0)|\psi^1 - \psi^0\rangle.$$

Fenchel-Young (in)equality:

$$\frac{1}{2}\mathbb{V}\mathrm{ar}_\rho(\psi^{1*} - \psi^{0*}) \leq \mathbb{V}\mathrm{ar}_{\frac{\mu^0+\mu^1}{2}}(\psi^1 - \psi^0).$$