Quantitative Stability of Barycenters in the Wasserstein Space

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Introduction

Wassertein Barycenters - Motivation

- Let $\mathcal{H}$ be a Hilbert space and $x_1, \dots, x_N \in \mathcal{H}$.
  Arithmetic mean of $x_1, \dots, x_N$ in $\mathcal{H}$:

$$\boxed{\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i = \arg\min_{\mu \in \mathcal{H}} \frac{1}{2N} \sum_{i=1}^{N} \|x_i - \mu\|_{\mathcal{H}}^2 \,.}$$

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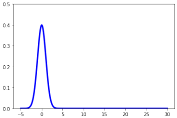
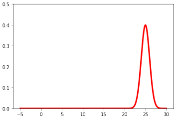
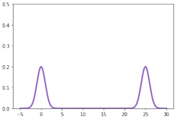
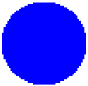
- Possibly ill-adapted in some cases:
  - Example: mean of two Gaussian p.d.f.

$\frac{1}{2}$  $+$ $\frac{1}{2}$  $=$ 

  - Example: mean of two shapes

$\frac{1}{2}$  $+$ $\frac{1}{2}$  $=$ 

Introduction

Wassertein Barycenters - Definition

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$$\boxed{\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i = \arg\min_{\mu \in \mathcal{H}} \frac{1}{2N} \sum_{i=1}^{N} \|x_i - \mu\|_{\mathcal{H}}^2 .}$$

- Let (M, d) be a complete metric space and $x_1, \ldots, x_N \in M$.
  Fréchet mean of $x_1, \ldots, x_N$ in (M, d):

$$\boxed{\bar{x} \in \arg\min_{\mu \in M} \frac{1}{2N} \sum_{i=1}^{N} d(x_i, \mu)^2.}$$

Introduction

Wassertein Barycenters - Definition

- Let $\Omega \subset B(0,R)$ be a compact subset of $\mathbb{R}^d$.
- $M = \mathcal{P}(\Omega)$ and $d = \mathrm{W}_2$:

$$\forall \alpha, \beta \in \mathcal{P}(\Omega), \mathrm{W}_2^2(\alpha, \beta) = \min_{\pi \in \Pi(\alpha,\beta)} \int_{\Omega \times \Omega} \|x - y\|^2 \, \mathrm{d}\pi(x, y),$$

  where $\Pi(\alpha, \beta) = \{\pi \in \mathcal{P}(\Omega \times \Omega) \mid (P_1)_\# \pi = \alpha \text{ and } (P_2)_\# \pi = \beta\}$ with $P_1(x, y) = x$ and $P_2(x, y) = y$.

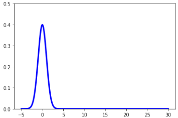
- Wasserstein barycenter of $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$:

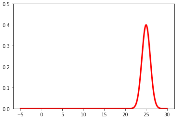
$$\boxed{\mu_{\rho_1,\ldots,\rho_N} \in \arg\min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2N} \sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu).}$$

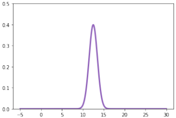
Introduction

Wassertein Barycenters - Examples

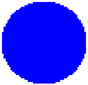
- Example: Wasserstein barycenter of two Gaussians

$\frac{1}{2}$ 

"$+$" $\frac{1}{2}$ 

$=$ 

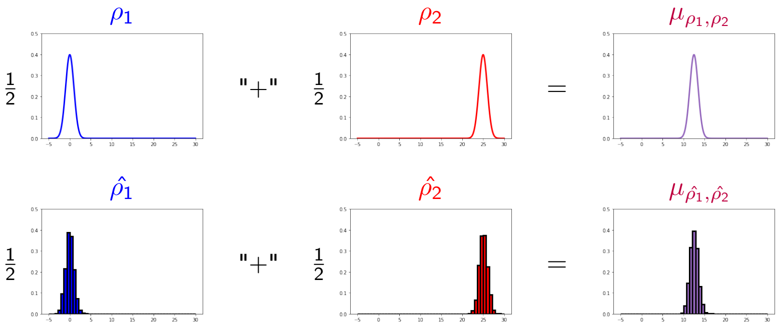
- Example: Wasserstein barycenter of two shapes

$\frac{1}{2}$  "$+$" $\frac{1}{2}$  $=$ 

Introduction

Wassertein Barycenters - Stability

- **In practice, measures are often accessible only through (noisy) approximations.**
- Example: Wasserstein barycenter of two (sampled) Gaussians

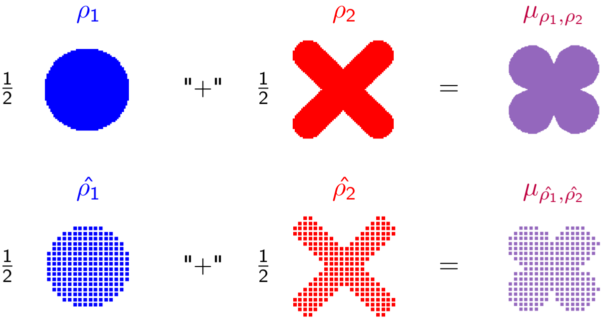


Can we get bounds on $\mathrm{W}_2(\mu_{\rho_1,\rho_2}, \mu_{\hat{\rho_1},\hat{\rho_2}})$ from bounds on $\mathrm{W}_2(\rho_1, \hat{\rho_1}), \mathrm{W}_2(\rho_2, \hat{\rho_2})$?

Introduction

Wassertein Barycenters - Stability

- **In practice, measures are often accessible only through (noisy) approximations.**
- Example: Wasserstein barycenter of two (downsampled) shapes



Can we get bounds on $\mathrm{W}_2(\mu_{\rho_1,\rho_2}, \mu_{\hat{\rho}_1,\hat{\rho}_2})$ from bounds on $\mathrm{W}_2(\rho_1, \hat{\rho}_1), \mathrm{W}_2(\rho_2, \hat{\rho}_2)$?

Stability of Wasserstein Barycenters

Some known results

- **Consistency** [LeGouic and Loubes (2017)]: If $\forall i, \mathrm{W}_2(\rho_i^n, \rho_i) \xrightarrow[n\to\infty]{} 0$, then $(\mu_{\rho_1^n,\ldots,\rho_N^n})_n$ is precompact and any cluster point belongs to

$$\arg\min_{\mu\in\mathcal{P}(\Omega)} \frac{1}{2N}\sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu).$$

Stability of Wasserstein Barycenters

Some known results

- **When** $d = 1$:
  W_2 is Hilbertian: $\mathrm{W}_2(\alpha, \beta) = \left\| F_\alpha^- - F_\beta^- \right\|_{\mathrm{L}^2([0,1])}$.

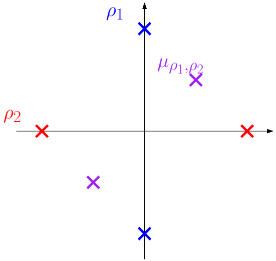
$$\implies \boxed{\mathrm{W}_2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\tilde{\rho}_1,\ldots,\tilde{\rho}_N}) \leq \frac{1}{N} \sum_{i=1}^{N} \mathrm{W}_2(\rho_i, \tilde{\rho}_i).}$$

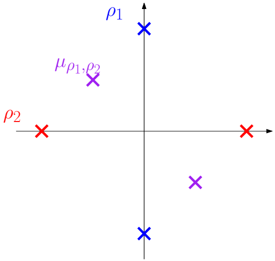
Quantitative results when $d > 1$?

Stability of Wasserstein Barycenters

Some negative results

- When $d > 1$, barycenter may not be unique:

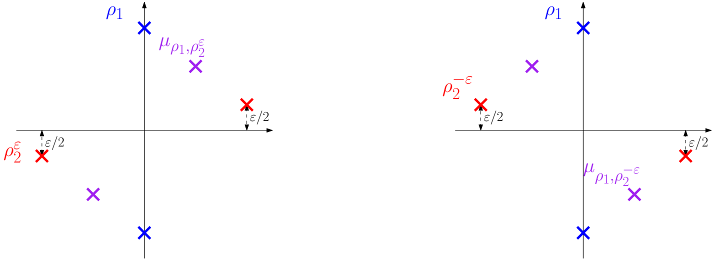




Stability of Wasserstein Barycenters

Some negative results

- **No quantitative stability is possible**:



$$\mathrm{W}_2(\rho_2^\varepsilon, \rho_2^{-\varepsilon}) = \varepsilon \text{ while } \mathrm{W}_2(\mu_{\rho_1,\rho_2^\varepsilon}, \mu_{\rho_1,\rho_2^{-\varepsilon}}) = 1.$$

Stability of Wasserstein Barycenters

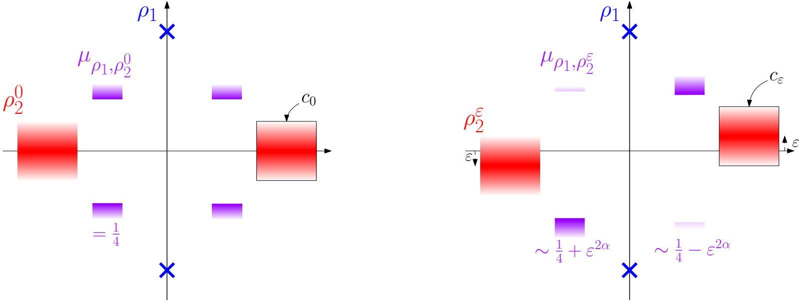
Some negative results

- Proposition 3.5 of [Agueh and Carlier (2011)]: **if one of the ρ_i's is absolutely continuous, the barycenter is unique**.

Stability of Wasserstein Barycenters

Some negative results

- Even with an a.c. marginal, α-**Hölder behaviour for any** $\alpha \in (0,1)$ **is possible**:



$$c^\varepsilon = [1 - \tfrac{a}{2}; 1 + \tfrac{a}{2}] \times [-\tfrac{a}{2} + \varepsilon; \tfrac{a}{2} + \varepsilon],$$

$$\rho_2^\varepsilon(x,y) \propto |y-\varepsilon|^{2\alpha-1} \chi_{c_\varepsilon}(x,y) + |y+\varepsilon|^{2\alpha-1} \chi_{c_{-\varepsilon}}(x,y).$$

$$\boxed{\mathrm{W}_2(\rho_2^0, \rho_2^\varepsilon) = \varepsilon \text{ while } \mathrm{W}_2(\mu_{\rho_1,\rho_2^0}, \mu_{\rho_1,\rho_2^\varepsilon}) \sim \varepsilon^\alpha.}$$

Stability of Wasserstein Barycenters

Assumption

- Dual formulation of OT: $\forall \rho, \mu \in \mathcal{P}(\Omega)$,

$$\min_{\pi \in \Pi(\rho,\mu)} \int_{\Omega \times \Omega} \|x - y\|^2 \,\mathrm{d}\pi(x, y) = M_2(\rho) + M_2(\mu) - 2 \min_{\psi \in \mathcal{C}(\Omega), \langle \psi | \mu \rangle = 0} \langle \psi^* | \rho \rangle,$$

with $M_2(\rho) = \langle \|\cdot\|^2 \, | \rho \rangle$, $M_2(\mu) = \langle \|\cdot\|^2 \, | \mu \rangle$, ψ^* the convex conjugate of ψ.

When ρ is a.c., an optimal ψ^* is a Brenier potential:

$$\mathrm{W}_2(\rho, \mu) = \|\nabla \psi^* - \mathrm{id}\|_{\mathrm{L}^2(\rho)} ,$$
$$\nabla \psi^* = T_{\rho \to \mu}.$$

Stability of Wasserstein Barycenters

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$$\min_{\pi \in \Pi(\rho,\mu)} \int_{\Omega\times\Omega} \|x-y\|^2 \, \mathrm{d}\pi(x,y) = M_2(\rho) + M_2(\mu) - 2 \min_{\psi \in \mathcal{C}(\Omega), \langle \psi | \mu \rangle = 0} \langle \psi^* | \rho \rangle.$$

- $\mathcal{K}_\rho : \psi \mapsto \langle \psi^* | \rho \rangle$ is in general **not globally strongly-convex**.

- **Main assumption:** local strong-convexity of OT problem from ρ_1:
$\exists c > 0$ s.t. $\forall \psi, \tilde{\psi} \in \mathcal{C}(\Omega)$,

$$c \mathbb{V}\mathrm{ar}_{\rho_1}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_{\rho_1}(\tilde{\psi}) - \mathcal{K}_{\rho_1}(\psi) - \langle \tilde{\psi} - \psi | \nabla \mathcal{K}_{\rho_1}(\psi) \rangle.$$

Stability of Wasserstein Barycenters

Strong-convexity assumption

Proposition (Carlier, D., Mérigot, 2022):

- *Let $\rho \in \mathcal{P}(\Omega)$ and assume*
  - *ρ is a.c.*
  - *$0 < m \leq \rho \leq M < +\infty$ on its support.*
  - *ρ **satisfies a Poincaré-Wirtinger inequality**: $\exists C_{PW} > 0$ s.t. $\forall f \in \mathcal{C}^1(\Omega)$, $\|f - \langle f|\rho\rangle\|_{\mathrm{L}^1(\rho)} \leq C_{PW} \|\nabla f\|_{\mathrm{L}^1(\rho)}$.*
  - *$\exists C_1, \dots, C_K \subset \Omega$ convex sets s.t. $\mathrm{spt}(\rho) = \bigcup_{k=1}^K C_k$.*
- *Then for all $\psi, \tilde{\psi} \in \mathcal{C}(\Omega)$,*

$$\boxed{c_\rho \mathbb{V}\mathrm{ar}_\rho(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi) - \langle \tilde{\psi} - \psi | \nabla \mathcal{K}_\rho(\psi) \rangle.}$$

where $c_\rho^{-1} = C_d R^2 \left(\frac{M}{m}\right)^2 K \left(1 + \frac{M^3 R^{3(d-1)} K^6 C_{PW}^3}{\varepsilon^6}\right)$ and C_d is a constant that depends only on d and $\varepsilon = \min\left(\min_{i,j | C_i \cap C_j \neq \emptyset} \rho(C_i \cap C_j), \min_i \rho\left(C_i \setminus \cup_{j \neq i} C_j\right)\right)$.

- Mainly derived from the **Brascamp-Lieb concentration inequality**.
- Used to get quantitative stability estimates for optimal transport maps w.r.t. target measures in [Delalande and Mérigot (2021)].

Stability of Wasserstein Barycenters

Main result

Theorem (Carlier, D., Mérigot, 2022):

- *Let* $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$ *and* $\tilde{\rho}_1, \ldots, \tilde{\rho}_N \in \mathcal{P}(\Omega)$.
- *Assume (A):*
  - ρ_1 *is a.c.*
  - $0 < m \leq \rho_1 \leq M < +\infty$ *on its support.*
  - $\mathcal{H}^{d-1}(\partial \mathrm{spt}(\rho_1)) \leq \mathrm{per} < +\infty$.
  - $\exists c > 0$ *s.t.* $\forall \psi, \tilde{\psi} \in \mathcal{C}(\Omega)$,

$$c \mathbb{V}\mathrm{ar}_{\rho_1}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_{\rho_1}(\tilde{\psi}) - \mathcal{K}_{\rho_1}(\psi) - \langle \tilde{\psi} - \psi | \nabla \mathcal{K}_{\rho_1}(\psi) \rangle.$$

- *Then*

$$\boxed{\mathrm{W}_2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\tilde{\rho}_1,\ldots,\tilde{\rho}_N}) \lesssim N^{1/4} \left(\frac{1}{N} \sum_{i=1}^{N} \mathrm{W}_2(\rho_i, \tilde{\rho}_i) \right)^{1/6},}$$

where $\lesssim$ *hides the multiplicative constant* $C_d (1+M)^{1/4} (1+R)^{\frac{d}{4}+1} (1 + \frac{M^{1/2} \mathrm{per}^{1/3}}{c^{1/6} m^{1/6}})$, *and* C_d *is a constant that depends only on* d.

Stability of Wasserstein Barycenters

Main result - Statistical consequence

- Let $\rho \in \mathcal{P}(\Omega)$ and $\hat{\rho}^n = \frac{1}{n}\sum_{j=1}^n \delta_{x_j}$ where $(x_j)_{1\leq j\leq n} \sim \rho^{\otimes n}$.
  Then [Fournier and Guillin (2015)]:

$$\mathbb{E}\mathrm{W}_2^2(\hat{\rho}^n, \rho) \leq C_d R^2 \begin{cases} n^{-1/2} & \text{if } d < 4, \\ n^{-1/2}\log(n) & \text{if } d = 4, \\ n^{-2/d} & \text{else.} \end{cases}$$

Corollary (Carlier, D., Mérigot, 2022):

- *Let $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$ with ρ_1 satisfying (A).*
- *$\forall i$, let $\hat{\rho}_i^n = \frac{1}{n}\sum_{j=1}^n \delta_{x_{i,j}}$ where $(x_{i,j})_{1\leq j\leq n} \sim \rho_i^{\otimes n}$.*
- *Then*

$$\mathbb{E}\mathrm{W}_2^2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\hat{\rho}_1^n,\ldots,\hat{\rho}_N^n}) \lesssim N^{1/2} \begin{cases} n^{-1/12} & \textit{if } d < 4, \\ n^{-1/12}\log(n)^{1/6} & \textit{if } d = 4, \\ n^{-1/(3d)} & \textit{else.} \end{cases}$$

Sketch of Proof

Two sub-problems

- Objective: show
$$W_2(\mu, \tilde{\mu}) \lesssim N^{1/4} \varepsilon^{1/6},$$
where $\mu = \mu_{\rho_1,\ldots,\rho_N}$, $\tilde{\mu} = \mu_{\tilde{\rho}_1,\ldots,\tilde{\rho}_N}$ and $\varepsilon = \frac{1}{N} \sum_{i=1}^{N} W_2(\rho_i, \tilde{\rho}_i)$.

- Assuming $\tilde{\rho}_1$ is a.c.:

$$\begin{aligned} W_2(\mu, \tilde{\mu}) &= W_2((T_{\rho_1 \to \mu})_\# \rho_1, (T_{\tilde{\rho}_1 \to \tilde{\mu}})_\# \tilde{\rho}_1) \\ &\leq \underbrace{W_2((T_{\rho_1 \to \mu})_\# \rho_1, (T_{\tilde{\rho}_1 \to \tilde{\mu}})_\# \rho_1)}_{(1)} + \underbrace{W_2((T_{\tilde{\rho}_1 \to \tilde{\mu}})_\# \rho_1, (T_{\tilde{\rho}_1 \to \tilde{\mu}})_\# \tilde{\rho}_1)}_{(2)}. \end{aligned}$$

Sketch of Proof

Bound on (1)

- Dual formulation of Wasserstein barycenter:

$$\underbrace{\min_{\mu\in\mathcal{P}(\Omega)} \frac{1}{2N}\sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i,\mu)}_{(\mathrm{P})_{\rho_1,\dots,\rho_N}} = \frac{1}{2N}\sum_{i=1}^{N} M_2(\rho_i) - \underbrace{\min_{(\psi_{\rho_i})_i\in\mathcal{F}} \frac{1}{N}\sum_{i=1}^{N} \mathcal{K}_{\rho_i}(\psi_{\rho_i})}_{(\mathrm{D})_{\rho_1,\dots,\rho_N}},$$

$$\text{with } \mathcal{F} = \{(\psi_{\rho_i})_{1\le i\le N} \in \mathcal{C}(\Omega)^N | \frac{1}{N}\sum_{i=1}^{N} \psi_{\rho_i}(\cdot) = \frac{\|\cdot\|^2}{2}\},$$

$$\mathcal{K}_{\rho_i} : \psi \mapsto \langle \psi^* | \rho_i \rangle.$$

- For an optimal $(\psi_{\rho_i})_{1\le i\le N}$, $\forall i, \psi_{\rho_i}$ is a Kantorovich potential:

$$\boxed{\mathrm{W}_2^2(\rho_i,\mu) = M_2(\rho_i) + M_2(\mu) - 2\left(\langle \psi_{\rho_i}^* | \rho_i\rangle + \langle \psi_{\rho_i} | \mu\rangle\right).}$$

- If ρ_i is a.c. : $\boxed{\nabla\psi_{\rho_i}^* = T_{\rho_i\to\mu}.}$

Sketch of Proof

Bound on (1)

- Dual formulation of Wasserstein barycenter:

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- Lipschitz behaviour of values:

$$\left|(\mathrm{P})_{\rho_1,\ldots,\rho_N} - (\mathrm{P})_{\tilde\rho_1,\ldots,\tilde\rho_N}\right| \le 3R\varepsilon,$$
$$\Longrightarrow \left|(\mathrm{D})_{\rho_1,\ldots,\rho_N} - (\mathrm{D})_{\tilde\rho_1,\ldots,\tilde\rho_N}\right| \le 4R\varepsilon.$$

- For $(\psi_{\rho_i})_i, (\psi_{\tilde\rho_i})_i$ minimizers of $(\mathrm{D})_{\rho_1,\ldots,\rho_N}, (\mathrm{D})_{\tilde\rho_1,\ldots,\tilde\rho_N}$, Kantorovich-Rubinstein implies then:

$$\left|\frac{1}{N}\sum_{i=1}^{N} \mathcal{K}_{\rho_i}(\psi_{\tilde\rho_i}) - \sum_{i=1}^{N}\frac{1}{N}\mathcal{K}_{\rho_i}(\psi_{\rho_i})\right| \le 5R\varepsilon.$$

Sketch of Proof

Bound on (1)

$$
\begin{aligned}
(1) &= \mathrm{W}_2((T_{\rho_1\to\mu})_{\#}\rho_1, (T_{\tilde\rho_1\to\tilde\mu})_{\#}\rho_1) \\
&\leq \| T_{\rho_1\to\mu} - T_{\tilde\rho_1\to\tilde\mu} \|_{\mathrm{L}^2(\rho_1)} \\
&= \left\| \nabla\psi^*_{\rho_1} - \nabla\psi^*_{\tilde\rho_1} \right\|_{\mathrm{L}^2(\rho_1)} && (\psi_{\rho_i})_i, (\psi_{\tilde\rho_i})_i \text{ solutions of } (\mathrm{D})_{(\rho_i)_i}, (\mathrm{D})_{(\tilde\rho_i)_i} \\
&\lesssim \mathbb{V}\mathrm{ar}_{\rho_1}(\psi^*_{\rho_1} - \psi^*_{\tilde\rho_1})^{1/6} && \text{Gagliardo-Nirenberg type ineq.} \\
&\lesssim (\mathcal{K}_{\rho_1}(\psi_{\tilde\rho_1}) - \mathcal{K}_{\rho_1}(\psi_{\rho_1}) + \langle \psi_{\tilde\rho_1} - \psi_{\rho_1} | \mu \rangle)^{1/6} && \text{Strong-convexity OT from } \rho_1. \\
&\leq \left(\sum_{i=1}^{N} \mathcal{K}_{\rho_i}(\psi_{\tilde\rho_i}) - \mathcal{K}_{\rho_i}(\psi_{\rho_i}) + \langle \psi_{\tilde\rho_i} - \psi_{\rho_i} | \mu \rangle \right)^{1/6} && \mathcal{K}_{\rho_i} \text{ convex.} \\
&= \left(\sum_{i=1}^{N} \mathcal{K}_{\rho_i}(\psi_{\tilde\rho_i}) - \mathcal{K}_{\rho_i}(\psi_{\rho_i}) \right)^{1/6} && \sum_i \psi_{\tilde\rho_i} = \sum_i \psi_{\rho_i} = N\frac{\|\cdot\|^2}{2}. \\
&\lesssim N^{1/6} \varepsilon^{1/6}.
\end{aligned}
$$

Sketch of Proof

Bound on (2)

$$\begin{aligned}(2) &= \mathrm{W}_2((T_{\tilde\rho_1\to\tilde\mu})_\#\rho_1, (T_{\tilde\rho_1\to\tilde\mu})_\#\tilde\rho_1)\\ &\le \| T_{\tilde\rho_1\to\tilde\mu} - T_{\tilde\rho_1\to\tilde\mu}\circ T_{\rho_1\to\tilde\rho_1}\|_{\mathrm{L}^2(\rho_1)}\\ &= \left\|\nabla\psi^*_{\tilde\rho_1} - \nabla\psi^*_{\tilde\rho_1}\circ T_{\rho_1\to\tilde\rho_1}\right\|_{\mathrm{L}^2(\rho_1)}\end{aligned}$$

In dimension $d = 1$, with $\Omega = [0,1]$:

- Denote $f = \nabla\psi^*_{\tilde\rho_1}$, with $f : [0,1] \to [0,1], f \nearrow$.
- Denote $\delta = \| T_{\rho_1\to\tilde\rho_1} - \mathrm{id}\|_{\mathrm{L}^\infty([0,1])}$. $\left(\delta = \| T_{\rho_1\to\tilde\rho_1} - \mathrm{id}\|_{\mathrm{L}^2([0,1])}\right)$
- Then:

$$(2)^2 \le \int_0^1 (f(x) - f(T_{\rho_1\to\tilde\rho_1}(x)))^2 \mathrm{d}\rho_1(x) \le \int_0^1 (f(x+\delta) - f(x-\delta))^2 \mathrm{d}\rho_1(x).$$

Sketch of Proof

Bound on (2)

$$(2)^2 \leq \int_0^1 (f(x+\delta) - f(x-\delta))^2 \mathrm{d}\rho_1(x).$$

- Denote $\mathcal{X}_\alpha = \{x \in [0,1] \quad | \quad \frac{f(x+\delta)-f(x-\delta)}{2\delta} \geq \delta^{-\alpha}\}$. Then,

$$\begin{aligned}
(2)^2 &\leq \int_0^1 (f(x+\delta) - f(x-\delta))^2 \mathrm{d}\rho_1(x) \\
&\leq \int_{(\mathcal{X}_\alpha)^c} 4\delta^{2(1-\alpha)} \mathrm{d}\rho_1 + \int_{\mathcal{X}_\alpha} (f(x+\delta) - f(x-\delta))^2 \mathrm{d}\rho_1(x) \\
&\leq 4\delta^{2(1-\alpha)} + \rho_1(\mathcal{X}_\alpha) \\
&\leq 4\delta^{2(1-\alpha)} + M\left|\mathcal{X}_\alpha\right|.
\end{aligned}$$

Sketch of Proof

Bound on (2)

- Denote $\mathcal{X}_\alpha = \{x \in [0,1] \quad | \quad \frac{f(x+\delta)-f(x-\delta)}{2\delta} \geq \delta^{-\alpha}\}$. Notice that

$$\delta^{-\alpha} |\mathcal{X}_\alpha| \leq \int_{\mathcal{X}_\alpha} \frac{f(x+\delta)-f(x-\delta)}{2\delta} \mathrm{d}x = \int_{\mathcal{X}_\alpha} \int_{|y-x|\leq\delta} \frac{f'(y)}{2\delta} \mathrm{d}y\mathrm{d}x$$
$$\leq \int_0^1 \int_{|y-x|\leq\delta} \frac{f'(y)}{2\delta} \mathrm{d}y\mathrm{d}x$$
$$\leq \int_{-\delta}^{1+\delta} f'(y)\mathrm{d}y \leq 1.$$

$$\implies \boxed{|\mathcal{X}_\alpha| \leq \delta^\alpha}.$$

- Then,

$$(2)^2 \lesssim 4\delta^{2(1-\alpha)} + M\delta^\alpha.$$

- Choosing $\alpha = \frac{2}{3}$ gives

$$\boxed{(2) \lesssim \delta^{1/3}.} \qquad ((2) \lesssim \delta^{1/4})$$

Sketch of Proof

Bound on (2)

$$
\begin{aligned}
(2) &= \mathrm{W}_2((T_{\tilde{\rho}_1\to\tilde{\mu}})_{\#}\rho_1, (T_{\tilde{\rho}_1\to\tilde{\mu}})_{\#}\tilde{\rho}_1) \\
&\le \| T_{\tilde{\rho}_1\to\tilde{\mu}} - T_{\tilde{\rho}_1\to\tilde{\mu}} \circ T_{\rho_1\to\tilde{\rho}_1} \|_{\mathrm{L}^2(\rho_1)} \\
&= \left\| \nabla\psi^*_{\tilde{\rho}_1} - \nabla\psi^*_{\tilde{\rho}_1} \circ T_{\rho_1\to\tilde{\rho}_1} \right\|_{\mathrm{L}^2(\rho_1)} \\
&\le C_d(1+M)(1+R)^{d+1}\mathrm{W}_2(\rho_1,\tilde{\rho}_1)^{1/4} \\
&\le C_d(1+M)(1+R)^{d+1} N^{1/4}\varepsilon^{1/4}.
\end{aligned}
$$

New result: stability of push-forward operation by gradient of convex Lipschitz function.

Sketch of Proof

Conclusion

$$
\begin{aligned}
\mathrm{W}_2(\mu, \tilde{\mu}) &\leq \underbrace{\mathrm{W}_2((T_{\rho_1 \to \mu})_{\#}\rho_1, (T_{\tilde{\rho}_1 \to \tilde{\mu}})_{\#}\rho_1)}_{(1)} + \underbrace{\mathrm{W}_2((T_{\tilde{\rho}_1 \to \tilde{\mu}})_{\#}\rho_1, (T_{\tilde{\rho}_1 \to \tilde{\mu}})_{\#}\tilde{\rho}_1)}_{(2)} \\
&\lesssim N^{1/6}\varepsilon^{1/6} + N^{1/4}\varepsilon^{1/4} \\
&\lesssim N^{1/4}\varepsilon^{1/6}.
\end{aligned}
$$

Stability - General result

Wassertein Barycenter - Extension of definition

- Wasserstein barycenter of $\rho_1, \dots, \rho_N \in \mathcal{P}(\Omega)$:

$$\mu_{\rho_1,\dots,\rho_N} \in \arg\min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2N} \sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu).$$

- Extension 1: weight ρ_i with $\alpha_i > 0$:

$$\boxed{\mu_{\alpha_1\rho_1,\dots,\alpha_N\rho_N} \in \arg\min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2\sum_i \alpha_i} \sum_{i=1}^{N} \alpha_i \mathrm{W}_2^2(\rho_i, \mu).}$$

- Extension 2: allow $N \to \infty$. Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$:

$$\boxed{\mu_{\mathbb{P}} \in \arg\min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2} \int_{\mathcal{P}(\Omega)} \mathrm{W}_2^2(\rho, \mu) \mathrm{d}\mathbb{P}(\rho).}$$

Stability - General result

Theorem (Carlier, D., Mérigot, 2022):

- *Let $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$. Assume that there exists $S_{\mathbb{P}} \subset \mathcal{P}(\Omega)$ such that $\mathbb{P}(S_{\mathbb{P}}) = \alpha_{\mathbb{P}} > 0$ and that $\forall \rho \in S_{\mathbb{P}}$,*
  - *ρ is a.c.*
  - *$0 < m \leq \rho \leq M < +\infty$ on its support.*
  - *$\mathcal{H}^{d-1}(\partial \mathrm{spt}(\rho)) \leq \mathrm{per} < +\infty$.*
  - *$\forall \psi, \tilde{\psi} \in \mathcal{C}(\Omega)$,*

$$c \mathbb{V}\mathrm{ar}_{\rho}(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_{\rho}(\tilde{\psi}) - \mathcal{K}_{\rho}(\psi) - \langle \tilde{\psi} - \psi | \nabla \mathcal{K}_{\rho}(\psi) \rangle.$$

- *Then*

$$\boxed{\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/4}} \mathcal{W}_1(\mathbb{P}, \mathbb{Q})^{1/6}.}$$

$$\forall \mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega)), \quad \mathcal{W}_1(\mathbb{P}, \mathbb{Q}) := \min_{\gamma \in \Pi(\mathbb{P}, \mathbb{Q})} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathrm{W}_2(\rho, \tilde{\rho}) \mathrm{d}\gamma(\rho, \tilde{\rho}).$$

Stability - General result

Statistics in the Wasserstein Space?

- Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ **regular enough** and $\mathbb{P}_m = \frac{1}{m}\sum_{i=1}^m \delta_{\rho_i}$ with $(\rho_i)_{1\leq i\leq m} \sim \mathbb{P}^{\otimes m}$. Then:

$$\boxed{\mathbb{E}\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/4}} \mathbb{E}\mathcal{W}_1(\mathbb{P}, \mathbb{P}_m)^{1/6}.}$$

- If **upper Wasserstein dimension of** $\mathbb{P} < s$ (Definition 4 of [Weed and Bach (2019)]) then,

$$\mathbb{E}\mathcal{W}_1(\mathbb{P}, \mathbb{P}_m) \lesssim m^{-1/s}.$$

- May be compared to:

[LeGouic et al. (2022), Corollary 2] Let $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$ and a barycenter $\mu_{\mathbb{P}}$ s.t. $\forall \rho \in \mathrm{spt}(\mathbb{P}), \rho = (\nabla\psi_{\mu_{\mathbb{P}}\to\rho})_{\#}\mu_{\mathbb{P}}$ with $\boxed{\alpha Id \preceq \mathrm{D}^2\psi_{\mu_{\mathbb{P}}\to\rho} \preceq \beta Id}$. Then if $\beta - \alpha < 1$, $\mu_{\mathbb{P}}$ is unique and

$$\boxed{\mathbb{E}\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{P}_m}) \leq \frac{4R}{\sqrt{1-\beta+\alpha}} m^{-1/2}.}$$

Thank you for your attention!

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Appendix

Local strong convexity of the Kantorovich functional $\mathcal{K}_\rho$

For ρ with convex support. Let $\mu^0, \mu^1 \in \mathcal{P}(\Omega)$ and for $k \in \{0, 1\}$,

$$\psi^k \in \arg \min_{\psi \in \mathcal{C}(\Omega), \langle \psi | \mu^k \rangle = 0} \mathcal{K}_\rho(\psi).$$

For $t \in [0, 1]$ denote $\psi^t = (1-t)\psi^0 + t\psi^1 = \psi^0 + tv$, and notice that:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_\rho(\psi^t) &= -\mathbb{E}_\rho v(\nabla \psi^{t*}), \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{K}_\rho(\psi^t) &= \mathbb{E}_\rho \langle \nabla v(\nabla \psi^{t*}) | \left(\mathrm{D}^2 \psi^t\right)^{-1} \nabla v(\nabla \psi^{t*}) \rangle. \end{aligned}$$

Brascamp-Lieb inequality:

$$\boxed{\mathbb{V}\mathrm{ar}_\rho(v(\nabla \psi^{t*})) \lesssim \mathbb{E}_\rho \langle \nabla v(\nabla \psi^{t*}) | \left(\mathrm{D}^2 \psi^t\right)^{-1} \nabla v(\nabla \psi^{t*}) \rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{K}_\rho(\psi^t).}$$

Appendix

Local strong convexity of the Kantorovich functional $\mathcal{K}_\rho$

Brascamp-Lieb inequality:

$$\mathbb{V}\mathrm{ar}_\rho(v(\nabla\psi^{t*})) \lesssim \mathbb{E}_\rho\langle \nabla v(\nabla\psi^{t*}) | \left(\mathrm{D}^2\psi^t\right)^{-1} \nabla v(\nabla\psi^{t*})\rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{K}_\rho(\psi^t).$$

$\int_0^1 \ldots \mathrm{d}t+$ concavity of $A \mapsto \det(A)^{1/d}$:

$$\mathbb{V}\mathrm{ar}_{\mu^0+\mu^1}(\psi^1-\psi^0) \lesssim \langle \nabla\mathcal{K}_\rho(\psi^1) - \nabla\mathcal{K}_\rho(\psi^0) | \psi^1-\psi^0\rangle.$$

Fenchel-Young (in)equality:

$$\mathbb{V}\mathrm{ar}_\rho(\psi^{1*}-\psi^{0*}) \leq \mathbb{V}\mathrm{ar}_{\mu^0+\mu^1}(\psi^1-\psi^0).$$

New Gagliardo-Niremberg type inequality for convex Lipschitz functions:

$$\|\nabla u - \nabla v\|^2_{\mathrm{L}^2(\mathcal{X})} \leq C_d \mathcal{H}^{d-1}(\partial\mathcal{X})^{2/3}(\|\nabla u\|_{\mathrm{L}^\infty(\mathcal{X})} + \|\nabla v\|_{\mathrm{L}^\infty(\mathcal{X})})^{4/3}\,\|u-v\|^{2/3}_{\mathrm{L}^2(\mathcal{X})}.$$

$$\left\| T_{\rho\to\mu^1} - T_{\rho\to\mu^0}\right\|_{\mathrm{L}^2(\rho)} = \left\|\nabla\psi^{1*} - \nabla\psi^{0*}\right\|_{\mathrm{L}^2(\rho)} \lesssim \mathbb{V}\mathrm{ar}_\rho(\psi^{1*}-\psi^{0*})^{1/6}$$

Appendix

Sketch of Proof - Bound on (2)

$$\begin{aligned}(2) &= \mathrm{W}_2((T_{\tilde\rho_1\to\tilde\mu})_\#\rho_1, (T_{\tilde\rho_1\to\tilde\mu})_\#\tilde{\rho_1}) \\ &\leq \| T_{\tilde\rho_1\to\tilde\mu} - T_{\tilde\rho_1\to\tilde\mu} \circ T_{\rho_1\to\tilde\rho_1} \|_{\mathrm{L}^2(\rho_1)} \\ &= \left\| \nabla\psi^*_{\tilde\rho_1} - \nabla\psi^*_{\tilde\rho_1} \circ T_{\rho_1\to\tilde\rho_1} \right\|_{\mathrm{L}^2(\rho_1)}\end{aligned}$$

Remark: Stability of push-forward by δ-isometry [Villani (2008)] not enough in general: L^2 bound on $T_{\rho_1\to\tilde\rho_1}$ instead of L^∞ (no 2δ-isometry), $\tilde\rho_1$ not a.c. (no OT map & push-forward: $(T_{\tilde\rho_1\to\tilde\mu})_\#\tilde{\rho_1}$ replaced by $\tilde\mu$).

In dimension $d \geq 1$:

- Denote $\delta = \| T_{\rho_1\to\tilde\rho_1} - \mathrm{id} \|_{\mathrm{L}^\infty(\Omega)}$. Then $\forall x \in \Omega$,

$$\left| \nabla\psi^*_{\tilde\rho_1}(x) - \nabla\psi^*_{\tilde\rho_1} \circ T_{\rho_1\to\tilde\rho_1}(x) \right| \leq \mathrm{diam}(\partial\psi^*_{\tilde\rho_1}(B(x,\delta))).$$

- Denote $\mathcal{X}_\alpha = \{x \in \Omega \quad | \quad \mathrm{diam}(\partial\psi^*_{\tilde\rho_1}(B(x,\delta))) \geq \delta^{-\alpha}\}$. Then,

$$(2)^2 \lesssim \delta^{-2\alpha} + |\mathcal{X}_\alpha|.$$

Appendix

Sketch of Proof - Bound on (2)

- Let $\mathcal{X}_\alpha^{4\delta}$ be a (4δ)-packing of $\mathcal{X}_\alpha$. Then $\forall x_i \in \mathcal{X}_\alpha^{4\delta}$:

$$\begin{aligned}
\delta^{-\alpha} &\leq \mathrm{diam}(\partial\psi^*_{\tilde{\rho}_1}(B(x_i,\delta))) \\
&\lesssim \delta^{-d} \left\|\nabla\psi^*_{\tilde{\rho}_1} - m_{B(x_i,4\delta)}\right\|_{\mathrm{L}^1(B(x_i,4\delta))} \quad \text{[Carlier, Eichinger, Kroshnin (2021)]} \\
&\lesssim \delta^{-d+1} \int_{B(x_i,4\delta)} \left\|\mathrm{D}^2\psi^*_{\tilde{\rho}_1}(x)\right\|_{1,1} \mathrm{d}x \quad \text{Poincaré-Wirtinger} \\
&\lesssim \delta^{-d+1} \int_{B(x_i,4\delta)} \Delta\psi^*_{\tilde{\rho}_1}(x)\mathrm{d}x \qquad \mathrm{D}^2\psi^*_{\tilde{\rho}_1} \text{ is symmetric p.s.d.}
\end{aligned}$$

- Therefore:

$$\begin{aligned}
|\mathcal{X}_\alpha| \lesssim \delta^d \left|\mathcal{X}_\alpha^{4\delta}\right| &\lesssim \delta^d \times \delta^{-d+1+\alpha} \int_{\Omega+B(0,4\delta)} \Delta\psi^*_{\tilde{\rho}_1}(x)\mathrm{d}x \\
&= \delta^{1+\alpha} \int_{\partial\Omega+B(0,4\delta)} \langle\nabla\psi^*_{\tilde{\rho}_1}(x)|n_x\rangle\mathrm{d}x \\
&\lesssim \delta^{1+\alpha}.
\end{aligned}$$

- Conclusion: $\boxed{(2) \lesssim \left(\delta^{-2\alpha} + \delta^{1+\alpha}\right)^{1/2} = \delta^{1/3}}$ for $\alpha = -1/3$.