Quantitative Stability in Quadratic Optimal Transport

Alex Delalande

EPFL

CIRM Workshop *PDE & Probability in interaction: functional inequalities, optimal transport and particle systems*

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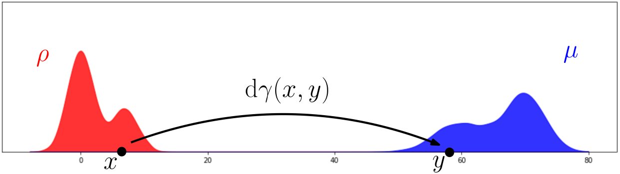
Introduction

Quadratic Optimal Transport problem (Monge, 1781; Kantorovich, 1942):

- Given $\rho, \mu \in \mathcal{P}_2(\mathbb{R}^d)$, solve

$$\boxed{\inf_{\gamma \in \Gamma(\rho,\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 \, \mathrm{d}\gamma(x, y),} \tag{KP}$$

where $\Gamma(\rho, \mu) = \{\gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) | \forall A \subset \Omega, \gamma(A \times \mathbb{R}^d) = \rho(A), \gamma(\mathbb{R}^d \times A) = \mu(A)\}$.



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- 2−***Wasserstein distance*** between ρ and μ:

$$\boxed{\mathrm{W}_2(\rho, \mu) := \left(\inf_{\gamma \in \Gamma(\rho,\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^2 \, \mathrm{d}\gamma(x, y) \right)^{1/2}.}$$

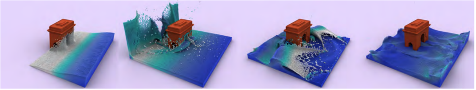
- 2−***Wasserstein space***: $(\mathcal{P}_2(\mathbb{R}^d), \mathrm{W}_2)$.

$\rightarrow$ Geodesic distance, interpolations, barycenters...

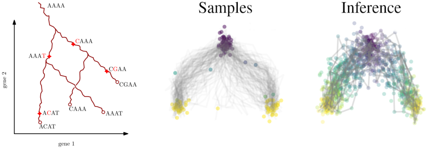
Introduction

- **Strong physical modeling power:**

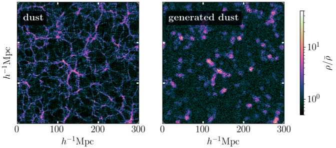
Euler equations: (de Goes et al., 2015)



Trajectory inference for single cell RNA-sequencing data: (Forrow et al., 2021; Chizat et al., 2022)



Cosmology: (Nikakhtar et al., 2023)



Quantum chemistry, **meteorology**, **economics**, **image processing**, **machine learning**...

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Is the quadratic optimal transport problem $\inf_{\gamma \in \Gamma(\rho,\mu)} \langle d^2 | \gamma \rangle$ ***well-posed*?**

1. **Existence of optimal γ? Verified.**
   (Kantorovich, 1942; Kellerer, 1984)

2. **Uniqueness of optimal γ? Not verified in general.**
   But well-studied and verified in many *particular* cases.

Theorem (Brenier, 1987): If ρ is **absolutely continuous**, then the optimal transport solution is **unique**. It is induced by a map $T : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $T_{\#}\rho = \mu$ characterized by $T = \nabla\phi$ with ϕ convex.

For μ a.c. and under additional assumptions, optimal ϕ solution of the **Monge-Ampère equation**:

$$\forall x \in \Omega, \quad \det(\mathrm{D}^2\phi(x))\mu(\nabla\phi(x)) = \rho(x).$$

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Example:

What is the *rate* of the convergence $\gamma_n \rightharpoonup \gamma$ in terms of the *rates* of convergence of $\rho_n \rightharpoonup \rho$ and $\mu_n \rightharpoonup \mu$?

$\rightarrow$ **Quantitative stability of optimal transport solutions?**

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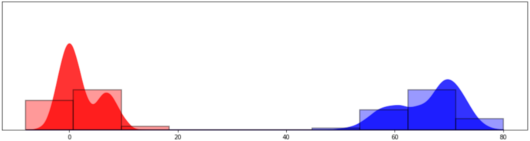

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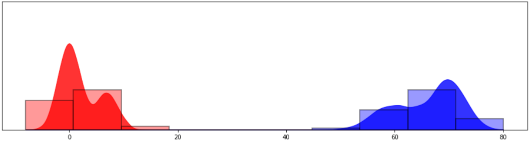

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1. Beyond the case $\Omega = \mathbb{R}$, no general quantitative stability result.
2. Convex optimization viewpoint: $\min_{\gamma \in \Gamma(\rho,\mu)} \langle d^2 | \gamma \rangle$.
   $\rightarrow$ **Strong/Uniform convexity?** *Linear program.*
3. Elliptic PDE viewpoint: $\det(\mathrm{D}^2\phi)\mu(\nabla\phi) = \rho.$
   $\rightarrow$ **Uniform ellipticity?** *Degenerate elliptic equation.*

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Part I.
Strong convexity of the dual quadratic optimal transport problem.

Part II.
Consequences: quantitative stability estimates for optimal transport maps, Wasserstein barycenters and entropic optimal transport.

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Dual problem

$$\min_{\gamma\in\Gamma(\rho,\mu)}\int \|x-y\|^2\,\mathrm{d}\gamma(x,y) = \int \|x\|^2\,\mathrm{d}\rho(x) + \int \|y\|^2\,\mathrm{d}\mu(y) - 2\underbrace{\max_{\gamma\in\Gamma(\rho,\mu)}\int\langle x|y\rangle\mathrm{d}\gamma(x,y)}_{:=\mathcal{T}(\rho,\mu)}.$$

- Get rid of constraint:

$$\mathcal{T}(\rho,\mu) = \max_{\gamma\geq 0}\int\langle\cdot|\cdot\rangle\mathrm{d}\gamma + \min_{\phi,\psi}\int\phi\mathrm{d}(\rho-\gamma_1) + \int\psi\mathrm{d}(\mu-\gamma_2)$$

- Swap max and min:

$$\begin{aligned}\mathcal{T}(\rho,\mu) &= \min_{\phi,\psi}\int\phi\mathrm{d}\rho + \int\psi\mathrm{d}\mu + \max_{\gamma\geq 0}\int(\langle\cdot|\cdot\rangle - \phi\oplus\psi)\mathrm{d}\gamma \\ &= \min_{\phi,\psi|\langle\cdot|\cdot\rangle\leq\phi\oplus\psi}\int\phi\mathrm{d}\rho + \int\psi\mathrm{d}\mu \\ &= \min_{\psi}\int\psi^*\mathrm{d}\rho + \int\psi\mathrm{d}\mu, \quad \text{where} \quad \psi^*(x) = \sup_y\langle x|y\rangle - \psi(y).\end{aligned}$$

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Definition: $\mathcal{K}_\rho : \psi \mapsto \int \psi^*\mathrm{d}\rho.$

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$$\boxed{\min_{\psi:\mathbb{R}^d\to\bar{\mathbb{R}}} \mathcal{K}_\rho(\psi) + \int \psi \mathrm{d}\mu.} \tag{DP}$$

- $\mathcal{K}_\rho$ **is convex.** First order condition in (DP):

$$\boxed{\psi_\mu \text{ minimizer in (DP)} \iff 0 \in \partial\mathcal{K}_\rho(\psi_\mu) + \mu \iff \psi_\mu \in (\partial\mathcal{K}_\rho)^{-1}(-\mu).}$$

Stability estimate for $\mu \mapsto \psi_\mu$.

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Strong/Uniform convexity estimate for $\psi \mapsto \mathcal{K}_\rho(\psi)$.

$F \in \mathcal{C}^2(\mathbb{R}^d)$ is strongly convex if there exists $\alpha > 0$ such that:

$$\forall x, y \in \mathbb{R}^d, t \in [0,1], \qquad \alpha\frac{t(1-t)}{2}\|x-y\|^2 \le (1-t)F(x) + tF(y) - F((1-t)x+ty),$$

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Strong convexity of the dual problem

- **Strong convexity of $\mathcal{K}_\rho : \psi \mapsto \int \psi^* \mathrm{d}\rho$?**
  1. *Strong convexity should work "up to additive constants":*

$$\boxed{\forall c \in \mathbb{R}, \quad \mathcal{K}_\rho(\psi + c) = \mathcal{K}_\rho(\psi) - c.}$$

  2. *Support of ρ should be connected:*

$$\nabla \psi^*_{\mu\#}\rho = \nabla \tilde{\psi}^*_{\mu\#}\rho = \mu.$$

$$\implies \psi_\mu, \tilde{\psi}_\mu \in \arg\min_\psi \mathcal{K}_\rho(\psi) + \langle \psi | \mu \rangle.$$

$$\implies \forall t \in [0,1],$$

$$\boxed{\mathcal{K}_\rho((1-t)\psi_\mu + t\tilde{\psi}_\mu) = (1-t)\mathcal{K}_\rho(\psi_\mu) + t\mathcal{K}_\rho(\tilde{\psi}_\mu).}$$

Assumption: Source ρ is **absolutely continuous** and satisfies a **Poincaré-Wirtinger inequality**: $\exists p \geq 1, C_{PW} \in (0, +\infty)$ s.t.

$$\forall f \in \mathcal{C}^1(\mathbb{R}^d), \quad \|f - \mathbb{E}_\rho f\|_{\mathrm{L}^p(\rho)} \leq C_{PW} \|\nabla f\|_{\mathrm{L}^p(\rho,\mathbb{R}^d)}.$$

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$$\boxed{\forall c \in \mathbb{R}, \quad \mathcal{K}_\rho(\psi + c) = \mathcal{K}_\rho(\psi) - c.}$$

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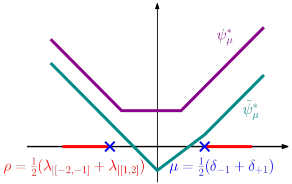

$$\nabla \psi_{\mu\#}^* \rho = \nabla \tilde{\psi}_{\mu\#}^* \rho = \mu.$$

$$\implies \psi_\mu, \tilde{\psi}_\mu \in \arg\min_\psi \mathcal{K}_\rho(\psi) + \langle \psi | \mu \rangle.$$

$$\implies \forall t \in [0,1],$$

$$\boxed{\mathcal{K}_\rho((1-t)\psi_\mu + t\tilde{\psi}_\mu) = (1-t)\mathcal{K}_\rho(\psi_\mu) + t\mathcal{K}_\rho(\tilde{\psi}_\mu).}$$

Assumption: Source ρ is **absolutely continuous** and satisfies a **Poincaré-Wirtinger inequality**: $\exists p \geq 1, C_{PW} \in (0, +\infty)$ s.t.

$$\forall f \in \mathcal{C}^1(\mathbb{R}^d), \quad \|f - \mathbb{E}_\rho f\|_{\mathrm{L}^p(\rho)} \leq C_{PW} \|\nabla f\|_{\mathrm{L}^p(\rho, \mathbb{R}^d)}.$$

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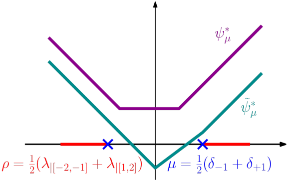

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- **"Subdifferential" of $\mathcal{K}_\rho$?** (Fenchel-Young) $\forall \psi, \tilde{\psi} : \mathbb{R}^d \to \bar{\mathbb{R}}$,

$$\langle \tilde{\psi} - \psi | -(\nabla \psi^*)_{\#}\rho \rangle \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi).$$

$\rightarrow$ Gap in this inequality?

- A known result:

Theorem (Ambrosio, Gigli, 2011):
Assume $\psi, \tilde{\psi} \in \mathcal{C}^1(\mathbb{R}^d)$, ψ is convex and $\tilde{\psi}$ **is α-strongly convex for some** $\alpha > 0$. Then,

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Strong assumption: $\tilde{\psi}$ is α-strongly convex $\iff$ $\nabla \tilde{\psi}^*$ is $\frac{1}{\alpha}$-Lipschitz continuous!

$\rightarrow$ Not satisfied in general.

$\rightarrow$ Implies that $\nabla \tilde{\psi}^*_{\#}\rho$ has a connected support.

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Strong convexity of the dual problem

Strong convexity estimate for the Kantorovich functional:

Theorem (D., Mérigot, 2021):

- Let $\mathcal{X} \subset \mathbb{R}^d$ compact convex, $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ with $0 < m_\rho \leq \rho \leq M_\rho$.
- Let $\mathcal{Y} = B(0, R_\mathcal{Y}) \subset \mathbb{R}^d$ compact and let $\mu, \nu \in \mathcal{P}(\mathcal{Y})$.

Then for ψ_μ, ψ_ν Kantorovich potentials between ρ and μ, ν,

$$\boxed{C_{d,\mathcal{X},\rho,\mathcal{Y}} \mathbb{V}\mathrm{ar}_{\frac{1}{2}(\mu+\nu)}(\psi_\nu - \psi_\mu) \leq \mathcal{K}_\rho(\psi_\nu) - \mathcal{K}_\rho(\psi_\mu) + \langle \psi_\nu - \psi_\mu | \mu \rangle,}$$

where $C_{d,\mathcal{X},\rho,\mathcal{Y}} = \left(e(d+1)2^{d-1} R_\mathcal{Y} \mathrm{diam}(\mathcal{X}) \frac{M_\rho^2}{m_\rho^2} \right)^{-1}$.

Strong convexity of the dual problem

$$\boxed{\mathbb{V}\mathrm{ar}_{\frac{1}{2}(\mu+\nu)}(\psi_\nu - \psi_\mu) \lesssim \mathcal{K}_\rho(\psi_\nu) - \mathcal{K}_\rho(\psi_\mu) + \langle \psi_\nu - \psi_\mu | \mu \rangle.}$$

- Remarks:
  1. Reference $\frac{1}{2}(\mu+\nu)$? Fenchel-Young (in)equality:

Proposition (D., Mérigot, 2021): $\frac{1}{2}\mathbb{V}\mathrm{ar}_\rho(\psi_\mu^* - \psi_\nu^*) \leq \mathbb{V}\mathrm{ar}_{\frac{1}{2}(\mu+\nu)}(\psi_\nu - \psi_\mu)$.

  2. Optimal exponents.
  3. Similar result with **non-compact targets**: main assumption is boundedness of ψ_μ^*, ψ_ν^* on $\mathcal{X}$. Satisfied if μ, ν have bounded moment of order $p > d$ (Morrey's inequality).
  4. $\mathrm{spt}(\rho) = \mathcal{X}$ convex? Can be relaxed.

Corollary (Carlier, D., Mérigot, 2022): If $\mathcal{X}$ is a **connected finite union of convex sets** s.t. ρ satisfies a L^1 **Poincaré-Wirtinger inequality**, a similar estimate holds.

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$$\boxed{\mathbb{V}\mathrm{ar}_{\frac{1}{2}(\mu^0+\mu^1)}(\psi^1 - \psi^0) \lesssim \mathcal{K}_\rho(\psi^1) - \mathcal{K}_\rho(\psi^0) + \langle \psi^1 - \psi^0 | \mu^0 \rangle.}$$

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Outline

Part I.
Strong convexity of the dual quadratic optimal transport problem.

Part II.
Consequences: quantitative stability estimates for optimal transport maps, Wasserstein barycenters and entropic optimal transport.

Quantitative stability of optimal transport maps

- **Motivation: (Approximated) Wasserstein geometry.**

1. **Theorem** (Brenier, 1987): If $\rho \in \mathcal{P}_{2,a.c.}(\mathbb{R}^d)$, $\exists! \rho$-a.e. $T_\mu : \mathbb{R}^d \to \mathbb{R}^d$ s.t. $(T_\mu)_\# \rho = \mu$, and $T_\mu = \nabla \phi_\mu$ with ϕ_μ **convex**.

$$\implies \boxed{\mathrm{W}_2(\rho, \mu) = \| T_\mu - \mathrm{id} \|_{\mathrm{L}^2(\rho, \mathbb{R}^d)} .}$$

2. Riemannian interpretation of $(\mathcal{P}_2(\mathbb{R}^d), \mathrm{W}_2)$ (Otto, 2001; Ambrosio, Gigli, Savaré, 2004):

	Riemannian geometry	Optimal transport
Point	$x \in M$	$\mu \in \mathcal{P}_2(\mathbb{R}^d)$
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3. *Linearized Optimal Transport:* $\boxed{\mathrm{W}_{2,\rho}(\mu, \nu) := \| T_\mu - T_\nu \|_{\mathrm{L}^2(\rho, \mathbb{R}^d)} .}$

Hilbertian distance used in image analysis (Wang et al., 2013) and other ML applications.

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How much does $\| T_\mu - T_\nu \|_{\mathrm{L}^2(\rho,\mathbb{R}^d)}$ distort $\mathrm{W}_2(\mu,\nu)$?

- Elementary results: $\mu \mapsto T_\mu$ is **continuous** and **reverse-Lipschitz**.

$$\mathrm{W}_2(\mu,\nu) \leq \| T_\mu - T_\nu \|_{\mathrm{L}^2(\rho,\mathbb{R}^d)} \,.$$

- Negative results: $\mu \mapsto T_\mu$ is **not better than $\frac{1}{2}$-Hölder**.

Theorem (Andoni, Naor, Neiman, 2018):
$(\mathcal{P}_2(\mathbb{R}^3), \mathrm{W}_2)$ does not admit a bi-Hölder embedding into any L^p space.

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Theorem (Ambrosio, Gigli, 2011): Let $\Omega \subset \mathbb{R}^d$ compact, $\rho \in \mathcal{P}_{a.c.}(\Omega)$ and $\mu, \nu \in \mathcal{P}(\Omega)$. Assume that T_μ is L-Lipschitz. Then,

$$\| T_\mu - T_\nu \|_{\mathrm{L}^2(\rho,\mathbb{R}^d)} \leq 2(\mathrm{diam}(\Omega) L)^{1/2} \mathrm{W}_1(\mu,\nu)^{1/2}.$$

Theorem (Berman, 2020): Let $\mathcal{X} \subset \mathbb{R}^d$ compact convex, $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ with $0 < m_\rho \leq \rho \leq M_\rho$. Let $\mathcal{Y} \subset \mathbb{R}^d$ bounded connected open with Lipschitz boundary and $\mu, \nu \in \mathcal{P}(\mathcal{Y})$. Then,

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Quantitative stability of optimal transport maps

Global & dimension-independent stability result (compact case):

Theorem (D., Mérigot, 2021):

- Let $\mathcal{X} \subset \mathbb{R}^d$ compact convex, $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ with $0 < m_\rho \leq \rho \leq M_\rho$.
- Let $\mathcal{Y} = B(0, R_\mathcal{Y}) \subset \mathbb{R}^d$ bounded and $\mu, \nu \in \mathcal{P}(\mathcal{Y})$.

Then the OT maps T_μ, T_ν between ρ and μ, ν satisfy

$$\| T_\mu - T_\nu \|_{\mathrm{L}^2(\rho, \mathbb{R}^d)} \leq C_{d,\rho,\mathcal{X},\mathcal{Y}} \mathrm{W}_1(\mu, \nu)^{1/6}.$$

- Remarks:
  1. Optimal Hölder exponent? $\frac{1}{6} < \frac{1}{2}$.
  2. The constant is explicit: $C_{d,\rho,\mathcal{X},\mathcal{Y}} \approx C_d \left(\frac{M_\rho}{m_\rho}\right)^3 \mathrm{diam}(\mathcal{X})^{d+1} R_\mathcal{Y}^2$.
  3. **Proof idea:** strong convexity of $\mathcal{K}_\rho$ and new Galgliardo-Nirenberg type inequality:

Proposition (D., Mérigot, 2021):
For $K \subset \mathbb{R}^d$ compact and $u, v : K \to \mathbb{R}$ Lipschitz convex,

$$\|\nabla u - \nabla v\|^2_{\mathrm{L}^2(K,\mathbb{R}^d)} \leq C_d \mathcal{H}^{d-1}(\partial K)^{\frac{2}{3}} (\mathrm{Lip}(u) + \mathrm{Lip}(v))^{\frac{4}{3}} \|u - v\|^{\frac{2}{3}}_{\mathrm{L}^2(K)}.$$

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Quantitative stability of optimal transport maps

Global stability result (non-compact case):

Theorem (Chazal, D., Mérigot, 2020/2021):

- Let $\mathcal{X} \subset \mathbb{R}^d$ compact convex, $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ with $0 < m_\rho \leq \rho \leq M_\rho$.
- Let $p > d$ and $p \geq 4$ and let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ with p-th moment upper bounded by $M_p < +\infty$.

Then the OT maps T_μ, T_ν between ρ and μ, ν satisfy

$$\| T_\mu - T_\nu \|_{\mathrm{L}^2(\rho, \mathbb{R}^d)} \leq C_{d,\rho,\mathcal{X},p,M_p} \mathrm{W}_1(\mu, \nu)^{\frac{p}{6p+16d}}.$$

- Remarks:
  1. Concerns all sub-Gaussian and sub-exponential probability measures.
  2. $\mu \mapsto T_\mu$ is a bi-Hölder embedding of $B_{\mathrm{W}_p}(\delta_0, M_p) \subset (\mathcal{P}_2(\mathbb{R}^d), \mathrm{W}_2)$ into $\mathrm{L}^2(\rho, \mathbb{R}^d)$.

Outline

Part I.
Strong convexity of the dual quadratic optimal transport problem.

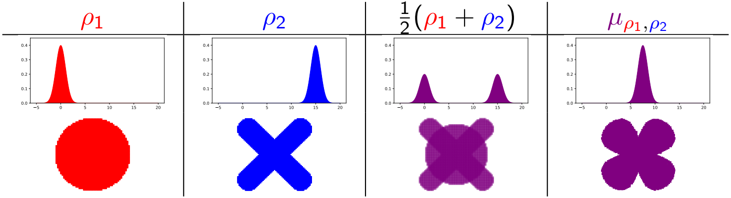
Part II.
Consequences: quantitative stability estimates for optimal transport maps, Wasserstein barycenters and entropic optimal transport.

Quantitative stability of Wasserstein barycenters

Definition: Let $\Omega \subset \mathbb{R}^d$ compact. *Wasserstein barycenter* of $\rho_1, \ldots, \rho_N \in \mathcal{P}(\Omega)$:

$$\mu_{\rho_1,\ldots,\rho_N} \in \arg \min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2N} \sum_{i=1}^{N} \mathrm{W}_2^2(\rho_i, \mu).$$

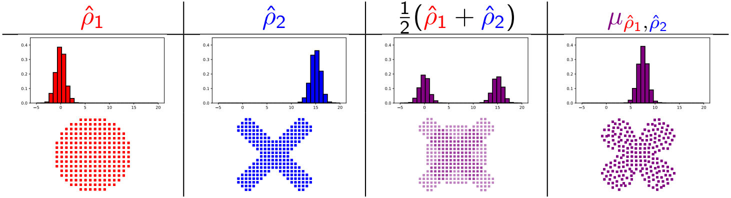
- Geometrically faithful "average" of probability measures:



Quantitative stability of Wasserstein barycenters

- Many applications, e.g. in
  1. Image processing (Rabin et al., 2011).
  2. Geometry processing (Solomon et al., 2015).
  3. Language processing (Colombo et al., 2021).

- ρ_1, ρ_2 often **not directly accessible**, but $\hat{\rho}_1$, $\hat{\rho}_2$ instead:



Can we bound $\mathrm{W}_2(\mu_{\hat{\rho}_1,\hat{\rho}_2}, \mu_{\rho_1,\rho_2})$ in terms of $\mathrm{W}_2(\hat{\rho}_1, \rho_1)$ and $\mathrm{W}_2(\hat{\rho}_2, \rho_2)$?

Quantitative stability of Wasserstein barycenters

- Known positive results:

Theorem (Le Gouic, Loubes, 2017): If $\forall i, \mathrm{W}_2(\rho_i^n, \rho_i) \xrightarrow[n\to\infty]{} 0$, then $(\mu_{\rho_1^n,\ldots,\rho_N^n})_n$ is pre-compact and any limit is a barycenter of $\rho_1, \ldots, \rho_N$.

Proposition: In dimension $d = 1$, $\mathrm{W}_2(\alpha, \beta) = \left\| F_\alpha^{-1} - F_\beta^{-1} \right\|_{\mathrm{L}^2([0,1])}$ so that:

$$\mathrm{W}_2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\tilde{\rho}_1,\ldots,\tilde{\rho}_N}) \leq \frac{1}{N} \sum_{i=1}^{N} \mathrm{W}_2(\rho_i, \tilde{\rho}_i).$$

Quantitative stability result in dimension $d \geq 2$?

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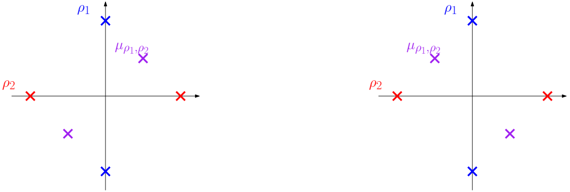
Quantitative stability of Wasserstein barycenters

- Negative results:

When $d \geq 2$, barycenter may not be unique:



Proposition (Agueh, Carlier, 2011): If one of the ρ_i's is absolutely continuous, the barycenter is unique.

Even with an a.c. marginal, α-Hölder behaviour for any $\alpha \in (0,1)$ is possible:

$W_2(\rho_2^0, \rho_2^\varepsilon) = \varepsilon$ while $W_2(\mu_{\rho_1,\rho_2^0}, \mu_{\rho_1,\rho_2^\varepsilon}) \sim \varepsilon^\alpha$.

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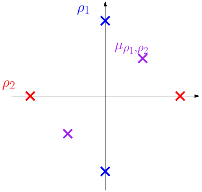

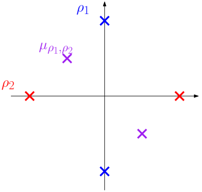

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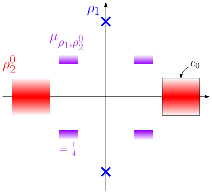

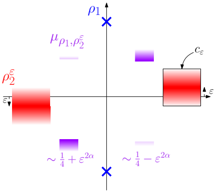

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Quantitative stability of Wasserstein barycenters

Hölder stability estimate for Wasserstein barycenters:

Theorem (Carlier, D., Mérigot, 2022):

- Let $\Omega = B(0, R_\Omega) \subset \mathbb{R}^d$ and $\rho_1, \ldots, \rho_N, \tilde{\rho}_1, \ldots, \tilde{\rho}_N \in \mathcal{P}(\Omega)$.
- Assume ρ_1 is a.c. with $0 < m_{\rho_1} \leq \rho_1 \leq M_{\rho_1}$ on $\mathcal{X}_1 = \mathrm{spt}(\rho_1)$ convex.

Then,
$$\boxed{\mathrm{W}_2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\tilde{\rho}_1,\ldots,\tilde{\rho}_N}) \leq C_{d,\Omega,\rho_1} \left(\sum_{i=1}^N \mathrm{W}_2(\rho_i, \tilde{\rho}_i) \right)^{1/6}.}$$

- Remarks:
  1. Optimal Hölder exponent?
  2. The constant is explicit: $C_{d,\Omega,\rho_1} \approx C_d (R_\Omega \mathrm{diam}(\mathcal{X}_1))^{d+1} \left(\frac{M_{\rho_1}}{m_{\rho_1}}\right)^{1/6}$.
  3. Main assumption on ρ_1: $\mathcal{K}_{\rho_1}$ should satisfy a strong convexity estimate.
  4. **Proof idea:** strong convexity of $\mathcal{K}_{\rho_1} \Rightarrow$ "strong convexity" of $\frac{1}{2}\mathrm{W}_2^2(\rho_1, \cdot)$:

Proposition (Carlier, D., Mérigot, 2022): Under the assumptions on ρ_1,

$$\forall \mu, \nu \in \mathcal{P}(\Omega), \quad \mathrm{W}_2^6(\mu,\nu) \lesssim \frac{1}{2}\mathrm{W}_2^2(\rho_1,\nu) - \frac{1}{2}\mathrm{W}_2^2(\rho_1,\mu) - \langle \frac{1}{2}\|\cdot\|^2 - \psi_{\rho_1\to\mu} | \nu - \mu \rangle.$$

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Quantitative stability of Wasserstein barycenters

- **Statistical consequence:**

Theorem (Fournier, Guillin, 2015):

- Let $\rho \in \mathcal{P}(\Omega)$ and $\hat{\rho}^n = \frac{1}{n}\sum_{j=1}^n \delta_{x_j}$ where $(x_j)_{1\leq j\leq n} \sim \rho^{\otimes n}$. Then:

$$\mathbb{E}\mathrm{W}_2^2(\hat{\rho}^n, \rho) \leq C_d R^2 \begin{cases} n^{-1/2} & \text{if } d < 4, \\ n^{-1/2}\log(n) & \text{if } d = 4, \\ n^{-2/d} & \text{else.} \end{cases}$$

Corollary (Carlier, D., Mérigot, 2022):

- Under the assumptions of the theorem, if $\forall i$, $\tilde{\rho}_i = \frac{1}{n}\sum_{j=1}^n \delta_{x_{i,j}}$ where $(x_{i,j})_{1\leq j\leq n} \sim \rho_i^{\otimes n}$, then

$$\mathbb{E}\mathrm{W}_2^2(\mu_{\rho_1,\ldots,\rho_N}, \mu_{\hat{\rho}_1^n,\ldots,\hat{\rho}_N^n}) \lesssim N^{1/3} \begin{cases} n^{-1/12} & \text{if } d < 4, \\ n^{-1/12}\log(n)^{1/6} & \text{if } d = 4, \\ n^{-1/(3d)} & \text{else.} \end{cases}$$

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Quantitative stability of entropic regularization

Entropic optimal transport: (Schrödinger, 1931; Léonard, 2014; Peyré and Cuturi, 2019)

- Primal and dual problems: for $\varepsilon > 0$,

$$\max_{\gamma \in \Gamma(\rho,\mu)} \int \langle x|y\rangle \mathrm{d}\gamma(x,y) - \varepsilon \mathrm{KL}(\gamma|\rho \otimes \mu) = \min_{\psi : \mathbb{R}^d \to \bar{\mathbb{R}}} \int \psi^{c,\varepsilon,\mu} \mathrm{d}\rho + \int \psi \mathrm{d}\mu + \varepsilon,$$

where $\psi^{c,\varepsilon,\mu}(\cdot) = \varepsilon \log \int e^{\frac{\langle \cdot|y\rangle - \psi(y)}{\varepsilon}} \mathrm{d}\mu(y)$.

$$\mathcal{K}_\rho^{\varepsilon,\mu} : \psi \mapsto \int \psi^{c,\varepsilon,\mu} \mathrm{d}\rho.$$

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Strong convexity of the dual entropic problem

Strong convexity estimate for the entropic Kantorovich functional:

Theorem (D., 2022):

- Let $\mathcal{X}$ compact convex, $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ with $0 < m_\rho \leq \rho \leq M_\rho$.
- Let $\mathcal{Y} = B(0, R_\mathcal{Y})$ compact and let $\mu \in \mathcal{P}(\mathcal{Y})$.

Then for ψ_μ an entropic potential between ρ and μ and any $v : \mathbb{R}^d \to \mathbb{R}$,

$$\boxed{C_{\mathcal{X},\rho,\mathcal{Y},\varepsilon} \mathbb{V}\mathrm{ar}_\mu(v) \leq \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{K}_\rho^{\varepsilon,\mu}(\psi_\mu + tv)\Big|_{t=0},}$$

where $C_{\mathcal{X},\rho,\mathcal{Y},\varepsilon} = \left(e^{R_\mathcal{Y} \mathrm{diam}(\mathcal{X})} \frac{M_\rho}{m_\rho} + \varepsilon\right)^{-1}$.

Remarks:

1. Allows to recover the *non-entropic* estimate when $\varepsilon \to 0$.
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- **Elements of proof:**
  1. Define $I : \psi \mapsto \log(\int_{\mathcal{X}} e^{-\psi^{c,\varepsilon,\mu}})$, and using the Prékopa-Leindler inequality, show

     $$\forall \varphi, \psi : \mathbb{R}^d \to \mathbb{R}, t \in [0,1], \quad I((1-t)\varphi + t\psi) \geq (1-t)I(\varphi) + tI(\psi).$$

  2. Noticing that I is $\mathcal{C}^2$, conclude with

     $$\frac{\mathrm{d}^2}{\mathrm{d}t^2} I(\psi_\mu + tv) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{K}_{\tilde{\rho}}^{\varepsilon,\mu}(\psi_\mu + tv) + Var_{\tilde{\mu}}(v) \leq 0.$$

- **Remark:** with $I : \psi \mapsto \log \int e^{-\psi^*}$, this argument recovers Brascamp-Lieb from Prékopa-Leindler.

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- **Remark:** with $I : \psi \mapsto \log \int e^{-\psi^*}$, this argument recovers Brascamp-Lieb from Prékopa-Leindler.

Quantitative stability of entropic regularization

- Denote $\psi^\varepsilon = \arg\min_\psi \mathcal{K}_\rho^{\varepsilon,\mu}(\psi) + \langle\psi|\mu\rangle + \varepsilon$. $\implies$ $\boxed{\nabla\mathcal{K}_\rho^{\varepsilon,\mu}(\psi^\varepsilon) + \mu = 0.}$
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Conclusion

- **Main ideas**:
  1. **Dual quadratic OT problem** may enjoy form of **strong/uniform convexity**.
  2. Strong convexity deduced from **Brascamp-Lieb/Prékopa-Leindler** inequalities $\rightarrow$ reinforces the link between OT and these functional inequalities.
  3. Allows to prove **stability results in OT**:
     - Guarantees for statistical/numerical approximation of OT maps and Wasserstein barycenters.
     - Partial embedding of $(\mathcal{P}_2(\mathbb{R}^d), \mathrm{W}_2)$ into $\mathrm{L}^2(\rho, \mathbb{R}^d)$.
     - Quantitative control of entropic regularization.

- **Open questions:**
  1. **Generalizations of strong convexity results:** source measure, ground cost?
  2. **Numerical consequences of strong convexity result:** Newton methods, interior point methods, etc.

Thank you for your attention!

Appendix

Quantitative stability of optimal transport maps

$$\boxed{\|T_\mu - T_\nu\|_{\mathrm{L}^2(\rho,\mathbb{R}^d)} \leq C_{d,\rho,\mathcal{X},\mathcal{Y}} \mathrm{W}_1(\mu,\nu)^{1/6}.}$$

- **Proof idea:** strong convexity of $\mathcal{K}_\rho$ and new Galgliardo-Nirenberg type inequality:

Proposition (D., Mérigot, 2021): For $K \subset \mathbb{R}^d$ compact and $u, v : K \to \mathbb{R}$ Lipschitz convex,

$$\|\nabla u - \nabla v\|^2_{\mathrm{L}^2(K,\mathbb{R}^d)} \leq C_d \mathcal{H}^{d-1}(\partial K)^{\frac{2}{3}} (\mathrm{Lip}(u) + \mathrm{Lip}(v))^{\frac{4}{3}} \|u - v\|^{\frac{2}{3}}_{\mathrm{L}^2(K)}.$$

Let ϕ_μ, ϕ_ν be convex Brenier potentials between ρ and μ, ν:

$$\begin{aligned}
\|T_\mu - T_\nu\|_{\mathrm{L}^2(\rho,\mathbb{R}^d)} &= \|\nabla\phi_\mu - \nabla\phi_\nu\|_{\mathrm{L}^2(\rho,\mathbb{R}^d)} \\
&\lesssim \mathbb{V}\mathrm{ar}_\rho(\phi_\mu - \phi_\nu)^{1/6} \\
&\leq \mathbb{V}\mathrm{ar}_{\frac{1}{2}(\mu+\nu)}(\phi^*_\mu - \phi^*_\nu)^{1/6} \\
&\lesssim (\mathcal{K}_\rho(\psi_\nu) - \mathcal{K}_\rho(\psi_\mu) + \langle \psi_\nu - \psi_\mu | \mu\rangle)^{1/6} \\
&\lesssim \mathrm{W}_1(\mu,\nu)^{1/6}.
\end{aligned}$$

Quantitative stability of Wasserstein barycenters

▶ **General result: infinite number of marginals**

Definition: Let $\Omega \subset \mathbb{R}^d$ compact. *Wasserstein barycenter* of $\mathbb{P} \in \mathcal{P}(\mathcal{P}(\Omega))$:

$$\mu_{\mathbb{P}} \in \arg\min_{\mu \in \mathcal{P}(\Omega)} \frac{1}{2} \int_{\mathcal{P}(\Omega)} \mathrm{W}_2^2(\rho, \mu) \mathrm{d}\mathbb{P}(\rho).$$

Theorem (Carlier, D., Mérigot, 2022):

▶ *Let* $\mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega))$ *and* $\exists S_{\mathbb{P}} \subset \mathcal{P}(\Omega)$ *s.t.* $\mathbb{P}(S_{\mathbb{P}}) = \alpha_{\mathbb{P}} > 0$ *and* $\forall \rho \in S_{\mathbb{P}}$,

1. ρ *is a.c.*
2. $0 < m \leq \rho \leq M < +\infty$ *on its support.*
3. $\mathcal{H}^{d-1}(\partial \mathrm{spt}(\rho)) \leq \mathrm{per} < +\infty$.
4. $\forall \psi, \tilde{\psi} \in \mathcal{C}(\Omega)$,

$$c \mathbb{V}\mathrm{ar}_\rho(\tilde{\psi}^* - \psi^*) \leq \mathcal{K}_\rho(\tilde{\psi}) - \mathcal{K}_\rho(\psi) - \langle \tilde{\psi} - \psi | \nabla \mathcal{K}_\rho(\psi) \rangle.$$

Then,

$$\mathrm{W}_2(\mu_{\mathbb{P}}, \mu_{\mathbb{Q}}) \lesssim \frac{1}{\alpha_{\mathbb{P}}^{1/4}} \mathcal{W}_1(\mathbb{P}, \mathbb{Q})^{1/6}.$$

$$\forall \mathbb{P}, \mathbb{Q} \in \mathcal{P}(\mathcal{P}(\Omega)), \quad \mathcal{W}_1(\mathbb{P}, \mathbb{Q}) := \min_{\gamma \in \Pi(\mathbb{P}, \mathbb{Q})} \int_{\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)} \mathrm{W}_2(\rho, \tilde{\rho}) \mathrm{d}\gamma(\rho, \tilde{\rho}).$$

Quantitative stability of entropic regularization

- Let $c(x,y) = \|x-y\|^2$.

	Classical OT	*Entropic* OT
Problem	$(\mathrm{P}_0)_{\rho,\mu} : \min_{\gamma\in\Gamma(\rho,\mu)} \langle c\|\gamma\rangle$	$(\mathrm{P}_\varepsilon)_{\rho,\mu} : \min_{\gamma\in\Gamma(\rho,\mu)} \langle c\|\gamma\rangle + \varepsilon \mathrm{KL}(\gamma\|\rho\otimes\mu)$
Computational complexity (n support points)	$\tilde{O}(n^3)$	$\tilde{O}(n^2/\varepsilon^2)$
Sample complexity $\mathbb{E}\left\|(\mathrm{P}.)_{\hat\rho^n,\hat\mu^n} - (\mathrm{P}.)_{\rho,\mu}\right\|$ ($\hat\rho^n = \frac{1}{n}\Sigma_i \delta_{x_i}, x_i \sim \rho$, $\hat\mu^n = \frac{1}{n}\Sigma_j \delta_{y_j}, y_j \sim \mu$)	$O(n^{-1/d})$	$\lesssim \frac{e^{1/\varepsilon}}{\varepsilon^{d/2}} n^{-1/2}$
Minimizer	γ^0	γ^ε
Geometry?	$\mathrm{W}_2(\rho,\mu) := \sqrt{\langle c\|\gamma^0\rangle}$ is a distance	$\mathrm{W}_{2,\varepsilon}(\rho,\mu) := \sqrt{\langle c\|\gamma^\varepsilon\rangle}$ is not a distance

Quantitative stability of entropic regularization

- **Semi-discrete setting:**
  - Let $\mathcal{X} \subset \mathbb{R}^d$ compact and $\rho \in \mathcal{P}_{a.c.}(\mathcal{X})$ absolutely continuous.
  - Let $\mathcal{Y} = \{y_1, \ldots, y_N\} \subset \mathbb{R}^d$ and $\mu = \sum_{i=1}^N \mu_i \delta_{y_i} \in \mathcal{P}(\mathcal{Y})$.

  $\rightarrow$ Natural framework in **statistics** and **numerical analysis**.

- A 2-dimensional example:

$\rho = \mathbb{1}_{[a,b]\times[c,d]}$ and $\mu = \sum_{i=1}^N \mu_i \delta_{y_i}$.

($\varepsilon = 0$)

$T_{\rho\to\mu}$ is piece-wise constant.

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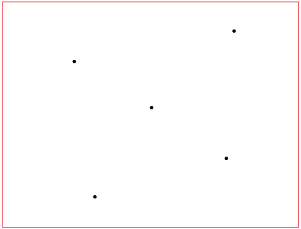

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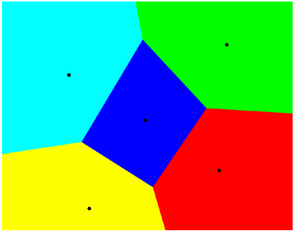

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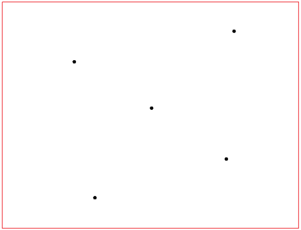

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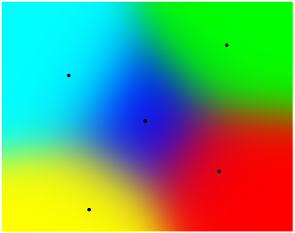

($\varepsilon = 7.5 \times 10^{-2}$)

γ^ε has same support as $\rho \otimes \mu$.

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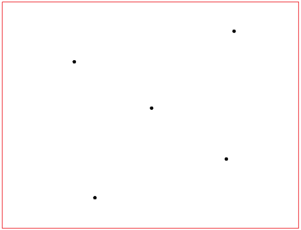

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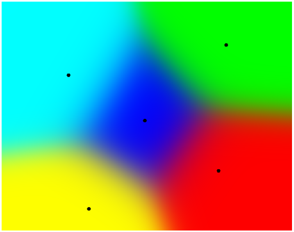

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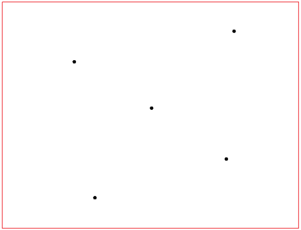

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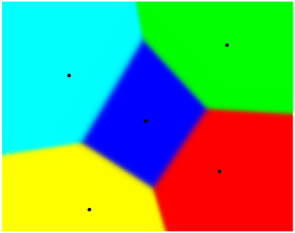

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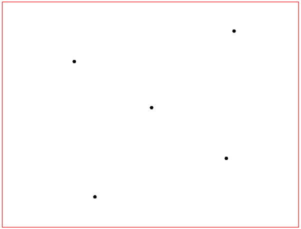

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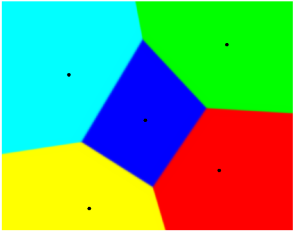

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